Problem 1 (10 points)

Consider the heat equation

$$u_t = \kappa u_{xx}, \qquad u(t=0,x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}$$

1. Write a MATLAB code that implements the explicit Euler scheme to solve the equation.
2. Write a MATLAB code that implements the implicit Euler scheme to solve the equation.
3. Compare the two methods in terms of stability and accuracy.

Problem 2 (10 points)

1. Consider the cubic nonlinear Schrödinger equation

   $$iu_t + u_{xx} + 2|u|^2 u = 0$$

   and use direct substitution to show that the following function is a solution:

   $$u(t,x) = \frac{\lambda}{\cosh(\lambda x)}\mathrm{e}^{i\lambda^2 t}$$

2. Show that the viscous Burger's equation $u_t + uu_x = \kappa u_{xx}$ can be linearized using the Cole-Hopf transfomation

   $$u = -2\kappa\frac{\phi_x}{\phi}$$

   meaning that u satisfies the above equation if ϕ satisfies a heat equation.

Problem 3 (10 points)

Consider the following partial differential equation (PDE) for $p = p(x,t)$

$$p_t = \partial_x(kxp) + \frac{\sigma^2}{2}p_{xx}, \qquad p(x,t=0) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$$

1. Show that $\int_{-\infty}^{\infty} p(x,t)dx$ is a conserved quantity.
2. Find a stationary solution of the above PDE.
3. Discretize the PDE in space and derive a system of (coupled) ordinary differential equations. Solve these equation using a 4-th order Runge-Kutta. Compare the solution to the stationary solution.