Problem 1 (10 points)

Consider a force field $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$ in the plane and a curve $\mathcal{C}$ given by $x(t) = a\cos(t)$ and $y(t) = b\sin(t)$ with $0 \leq t \leq \pi$ and positive constants a and b.

1. Compute $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ using the definition of the line integral. Does $\vec{F}$ have a potential?
2. Let a different force field be given by $\vec{G} = 2xy\vec{i} + (x^2 - y^2)\vec{j}$. Does $\vec{G}$ have a potential? If yes, find the potential of $\vec{G}$.
3. Calculate $W = \int_{\gamma} \vec{G} \cdot d\vec{r}$ where γ is the semi-circle connecting the points $(0,-1)$ and $(0,1)$. Use again the definition of the line integral.
4. Calculate W again by using a line integral connecting the above points. Can you think of a third way to get W easily?

Problem 2 (10 points)

1. Find the Laurent series for the following function

$$f(z) = \frac{1}{z^2 - (2+i)z + 2i}$$

2. The complex Bessel function $J_n(z)$ of order n can be defined for an integer n by

$$\mathrm{e}^{z(w-1/w)/2} = \sum_{n=-\infty}^{\infty} J_n(z) w^n$$

   Hence, $J_n(z)$ is the coefficient of w_n in the Laurent series of $\exp(z(w-1/w)/2)$, as a function of w, about 0.

   (a) Use the integral formula for Laurent coefficients to show that

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta$$

   (b) Write

$$\mathrm{e}^{z(w-1/w)/2} = \mathrm{e}^{zw/2}\mathrm{e}^{-z/(2w)}$$

   and multiply the series expansions of these function about 0 to obtain

$$J_n(z) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(n+k)!} \left(\frac{z}{2}\right)^{n+2k}$$

Problem 3 (10 points)

Suppose that $f = u + iv$ is a complex-valued function that is continuous and has continuous first partial derivatives. Use a complex parametrization $z = \gamma(t) = x(t) + iy(t)$ to show that

$$\int_\gamma f(z)\,dz = \int_\gamma u\,dx - vdy + i\int_\gamma vdx + udy$$

Assume now that γ is closed and that f is analytic. Use Green's Theorem and the Cauchy-Riemann relations to show that

$$\int_\gamma f(z)dz = 0$$