Problem 1 (10 points)

Consider the Gamma function $\Gamma(x)$ and the Beta function $B(x)$ defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} dt, \qquad B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

1. Show that $\Gamma(x+1) = x\Gamma(x)$.
2. Show that $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.
3. A particle of mass m moves in a one-dimensional force field with a potential $V(x) = A|x|^n$, $A, n > 0$. Sketch the potential and state the range of energy and position within which bounded oscillations can occur. Determine the period of oscillation as a function of the energy E.

Problem 2 (10 points)

Consider the following eigenvalue problem:

$$y^{(4)}(x) = \lambda y(x), \quad y(0) = y'(0) = 0, \qquad y''(L) = y'''(L) = 0$$

1. Find an equation for the eigenmodes of the system.
2. Check your result for the first eigenmode using a Chebyshev-differentiation matrix.

Problem 3 (10 points)

Consider the Chebyshev-polynomials T_n and use the recursion formula

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$$

and find the corresponding generating function $g(x,t)$.