Problem 1 (10 points)

1. Consider the example of the construction of a Cauchy sequence in $C([0,1)]$ given in the book on page 81 (4.10) and show that, indeed, $\|f_n - f_m\| \to 0$ as $m, n \to \infty$.
2. Are there functions that are in L^1 but not in L^2? Are there functions that are in L^2 but not in L^1?

Problem 2 (10 points)

Assume that A is a self-adjoint operator and that the underlying inner product space is finite-dimensional. Consider the eigenvalue problem $Af = \lambda f$.

1. Show that the eigenvalues are real.
2. Show that eigenvectors to different eigenvalues are orthogonal.
3. Assume that B is also self-adjoint. Assume for simplicity that all eigenspaces of A and B have the dimension 1. Show that A and B commute if and only if they have the same eigenspaces.

Problem 3 (10 points)

Consider the following eigenvalue problem:

$$L\psi = \lambda\psi, \qquad L = -\frac{d^2}{dx^2} + x^2$$

1. Show that L is self-adjoint.
2. Show that $\psi_1(x) = \alpha x \exp(-x^2/2)$ is an eigenfunction. What is the corresponding eigenvalue λ_1? Find α such that $\|\psi_1\| = 1$.
3. Use now a discretized version of the above problem in Matlab in order to find numerically the above eigenfunction ψ_1 when you prescribe λ_1. Compare the numerical solution to the analytical solution. Attach both the corresponding graph and a print-out of your code to your solution.