Math Methods

Lecture 10

Method of stationary phase

Consider ($\lambda \gg 1$) $\quad I(\lambda) = \int_a^b f(x)\, e^{i\lambda \phi(x)}\, dx$

Assume that $\phi'(x_s) = 0$. The major contribution to $I(\lambda)$ comes from a small neighborhood around x_s: Write

$$I(\lambda) = e^{i\lambda\phi(x_s)} \int_a^b f(x)\, e^{i\lambda(\phi(x)-\phi(x_s))}\, dx$$

highly oscillatory

$$\phi(x) = \phi(x_s) + \frac{1}{2}\phi''(x_s)(x-x_s)^2 + \cdots \quad , \xi = x - x_s$$

$$I(\lambda) \approx e^{i\lambda\phi(x_s)} \int_{-\infty}^{\infty} f(x_s)\, e^{i\frac{\lambda}{2}\phi''(x_s)\xi^2}\, d\xi \qquad (\phi''(x_s) > 0)$$

$$\boxed{I(\lambda) \approx f(x_s)\, e^{i\lambda\phi(x_s)} \sqrt{\frac{2\pi i}{\lambda \phi''(x_s)}}}$$

Example :

Bessel function: $$J_n(\lambda) = \int_0^1 \cos(n\pi x - \lambda \sin \pi x)\, dx$$

$$= Re\left(\int_0^1 \underset{\uparrow\atop f(x)}{e^{n\pi i x}} \underset{\uparrow\atop e^{i\lambda\phi(x)}}{e^{-i\lambda \sin \pi x}}\, dx \right)$$

$\phi(x) = -\sin(\pi x)$

and $\phi'(x) = -\pi \cos(\pi x)$ is 0 at $x_s = \frac{1}{2}$

$\phi(x_s) = -1 \quad \phi''(x_s) = \pi^2 \sin(\pi x_s) = \pi^2$

$f(x_s) = e^{n\pi i/2}$

$$\Rightarrow J_n(\lambda) \approx Re\left(e^{n\pi i/2} e^{-i\lambda} \sqrt{\frac{2\pi i}{\lambda \pi^2}} \right)$$

Now $\sqrt{\frac{2\pi i}{\lambda\pi^2}} = \sqrt{\frac{2}{\lambda\pi}}\, e^{i\pi/4}$

$$J_n(\lambda) \approx \sqrt{\frac{2}{\lambda\pi}}\, Re\left(e^{-i\left(\lambda - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \right)$$

$$\boxed{J_n(\lambda) \approx \sqrt{\frac{2}{\lambda\pi}} \cos\left(\lambda - \frac{n\pi}{2} - \frac{\pi}{4}\right)}$$

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Matlab:

```
z = linspace (2, 30, 1000);
                              n
y= sqrt (2./(z*pi)).* cos(z - 2*pi/2 - pi/4);
plot (z, besselj (2,z), z, y)
```

Method of steepest descent

How can we evaluate $\int_{-\infty}^{\infty} e^{iax^2}\,dx$?

We know $\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$. We will show that

$$J = \int_{-\infty}^{\infty} e^{-iax^2}\,dx = \sqrt{\frac{\pi}{ia}} = \sqrt{\frac{\pi}{a}}\; e^{-\frac{i\pi}{4}}$$

$$e^{-iax^2} = \cos(ax^2) - i\sin(ax^2)$$

most contributions will come from region around zero.

$$e^{-iaz^2} \begin{cases} \text{along} & z = L - iw \quad (w>0) \\ \text{and} & z = -L + iw \quad (w>0) \\ \text{small} & \end{cases}$$

$$(L - iw)^2 = L^2 - w^2 - 2iwL$$

$$-iaz^2 = -ia(L^2 - w^2) - 2awL \leftarrow \text{decaying}$$

Therefore: we can integrate along this line

$$z = (1-i)\,t \qquad 1-i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$$

$$dz = (1-i)\,dt \qquad z^2 = -2it^2$$

$$\Rightarrow \quad J = \left(\int_{-\infty}^{\infty} e^{-2at^2}\, dt\right)\cdot \sqrt{2}\, e^{-i\frac{\pi}{4}} = \sqrt{\frac{\pi}{2a}}\,\sqrt{2}\, e^{-i\frac{\pi}{4}}$$

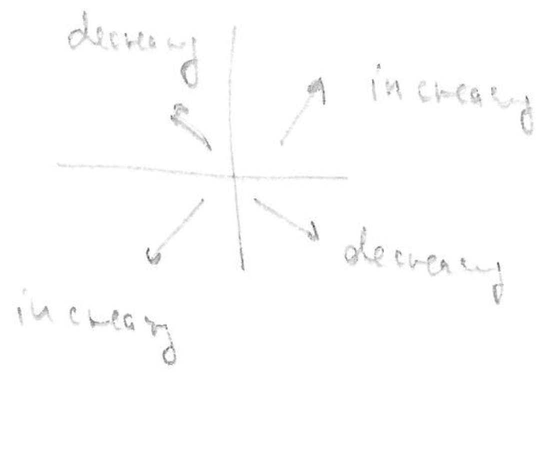
$$\Rightarrow \quad J = \sqrt{\frac{\pi}{a}}\, e^{-i\frac{\pi}{4}} \quad \text{as claimed}$$

Idea: distort integrand in a way that it decays as fast as possible away from the stationary point.

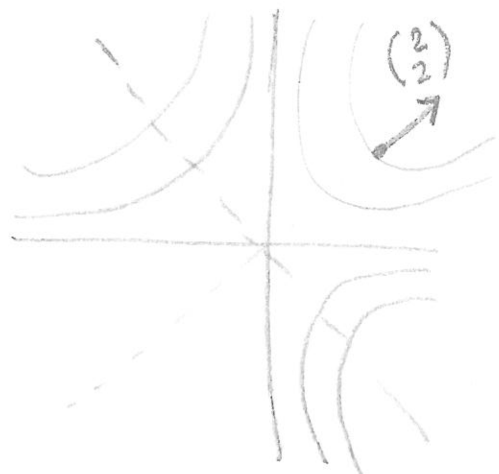
Consider e^{-iaz^2} and $z = x+iy$

$$\Rightarrow -iaz^2 = -ia\,(x^2-y^2+2ixy) = -ia(x^2-y^2) + 2xy$$

$\Rightarrow$



Steepest descent lines: • perpendicular to lines of const. Re $f(z)$



• phase is const.

$\Rightarrow$ all contributions are coherent and add up.

General: write $e^{\lambda u(x,y)} \, e^{i\lambda v(x,y)}$

For a point $z_0 = (x_0, y_0)$, we know that $\nabla u \big|_{z_0}$ defines the direction of the steepest ascent, $-\nabla u \big|_{z_0}$ the direction of the steepest descent.

$z = x + iy \Rightarrow -iaz^2 = 2xy - ia(x^2 - y^2)$

$$\Rightarrow \nabla u = 2 \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{e.g. } \nabla u \Big|_{(1,1)} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

General saddle point of order $N-1$:

$$p'(z_0), \dots, p^{(N-1)}(z_0) = 0, \quad p^{(N)}(z_0) \neq 0$$

$$\Rightarrow \quad p(z) = p(z_0) + \frac{(z-z_0)^N}{N!} p^{(N)}(z_0) + \dots$$

$$z = z_0 + \rho e^{i\theta}, \quad p^{(N)}(z_0) = a e^{i\alpha}$$

$$\Rightarrow \quad p(z) - p(z_0) \sim \frac{\rho^N a e^{i(N\theta + \alpha)}}{N!}$$

Curves of steepest ascent / descent are given by $\operatorname{Im}(p(z) - p(z_0)) = 0 \Rightarrow \sin(N\theta + \alpha) = 0$

hence $N\theta + \alpha = k\pi, \; k \in \mathbb{Z}$

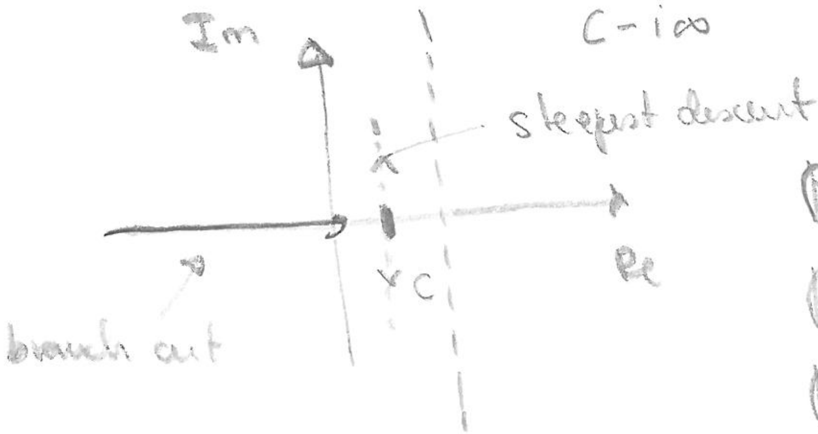
$$\Rightarrow \quad p(z) - p(z_0) = u(x,y) - u(x_0,y_0) \sim \frac{\rho^N a}{N!} \cos(N\theta + \alpha)$$

and we want this to be <u>negative</u>.

$$\Rightarrow \theta = -\frac{\alpha}{N} + (2k+1)\frac{\pi}{N} \Rightarrow \cos(N\theta+\alpha) = -1.$$

This defines the curves of steepest descent.

Ex. $$I(\lambda) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\lambda(at-\sqrt{t})}}{t}\,dt \qquad a, c > 0$$



$$p(t) = at - \sqrt{t}$$
$$p'(t) = a - \frac{1}{2}t^{-1/2} = a - \frac{1}{2\sqrt{t}}$$
$$p''(t) = \frac{1}{4}t^{-3/2}$$

Saddle: $p'(t) = 0 \Rightarrow t = t_0 = \frac{1}{4a^2}$, $p''(t_0) = 2a^3$

Here, $N=2$, $p''(t_0) = 2a^3$ does not have a phase, here directions of steepest descent are

$\theta = \frac{\pi}{2}, -\frac{\pi}{2}$

$\Rightarrow$ Basic approximation straight line in direction of steepest descent

$$t = \frac{1}{4a^2} + \underset{\text{small}}{T}i \quad , \text{ we know derivative vanishes...}$$

$$\Rightarrow at - \sqrt{t} = \underbrace{\frac{a}{4a^2} - \sqrt{\frac{1}{4a^2}}}_{(T=0)} - \frac{1}{2}\underbrace{2a^3}_{\text{second derivative}} T^2$$

$$= - \frac{1}{4a} - a^3 T^2 \quad \text{and} \quad \frac{1}{t} \approx 4a^2$$

$$\Rightarrow I(\lambda) = \frac{1}{2\pi i}\int \frac{e^{\lambda(at - \sqrt{t})}}{t}\, dt$$

$$\approx \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{\lambda}{4a}}\, e^{-a^3T^2\lambda}\, 4a^2 i\, dT$$

$$= e^{-\frac{\lambda}{4a}}\, \frac{2a^2}{\pi}\int_{-\infty}^{\infty} e^{-a^3T^2\lambda}\, dT = e^{-\frac{\lambda}{4a}}\, \frac{2a^2}{\pi}\sqrt{\frac{\pi}{a^3\lambda}}$$

$$= 2\sqrt{\frac{a}{\pi\lambda}}\, e^{-\lambda/4a}$$

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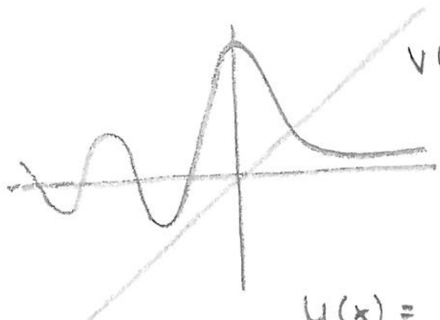
Airy's differential equation

$(*)\quad y'' = xy$ e.g. linear potential in Schrödinger eq.



$V(x)=x$ $\quad H\psi = (-\partial_x^2 + x)\psi = E\psi$ with $E=0$

Idea: Solve $(*)$ using a contour integral

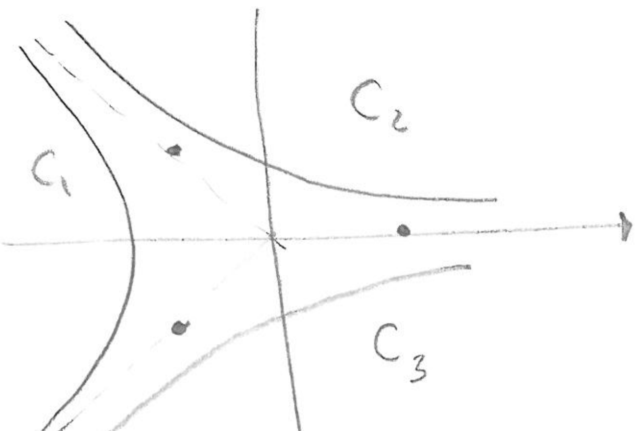
$$y(x) = \int_C e^{xt} g(t)\, dt \,. \quad \text{Let } L = \partial_x^2 - x$$

$$\text{So } Ly = \int_C (t^2 - x)\, e^{xt} g(t)\, dt \overset{!}{=} 0 \overset{!}{=} \int_C \frac{\partial Q}{\partial t}\, dt$$

with $Q = Q(x,t)$ and C such that C is either closed or Q vanishes at both ends of the contour.

Choose $Q(x,t) = e^{xt} e^{-t^3/3}$ and contours such that

$$\left[e^{xt} e^{-t^3/3} \right]_C = 0 \,, \quad \text{works for } \begin{array}{l} t \to \infty \\ t \to e^{2\pi i/3}\infty \\ t \to e^{-2\pi i/3}\infty \end{array}$$



$$Ai(x) = \frac{1}{2\pi i} \int_{C_1} e^{xt - \frac{1}{3}t^3}\, dt$$

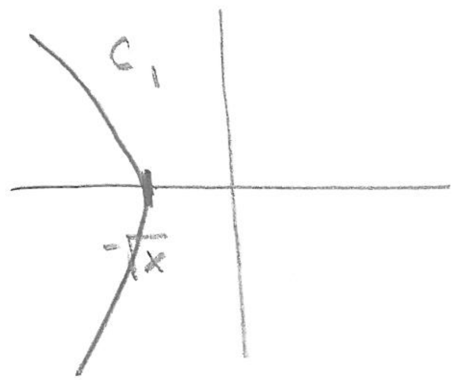
$$(**)\quad = \frac{1}{\pi} \int_0^\infty \cos\left(xs + \frac{1}{3}s^3\right) ds$$

<u>Note:</u> (**) is an exercise

<u>Asymptotics:</u> Use steepest descent.

$\boxed{x \gg 0}$ $\varphi(t) = xt - \frac{1}{3}t^3$, $\varphi'(t) = x - t^2 \Rightarrow t = \pm\sqrt{x}$

and we choose $t = -\sqrt{x}$



Close to $t = -\sqrt{x}$ we have

$t = -\sqrt{x} + \xi$ and

$$\varphi(t) \approx \varphi(-\sqrt{x}) + \frac{1}{2}\varphi''(-\sqrt{x})\,\xi^2$$

$$= -\frac{2}{3}x^{3/2} + \sqrt{x}\,\xi^2$$ and we choose a vertical contour

$\xi = is$ such that φ has a maximum at $s = 0$.

$$\Rightarrow \quad Ai(x) \approx \frac{1}{2\pi i}\, e^{-\frac{2}{3}x^{3/2}} \int_{-i\infty}^{i\infty} e^{\sqrt{x}\,\xi^2}\, d\xi$$

$$= \frac{1}{2\pi}\, e^{-\frac{2}{3}x^{3/2}} \int_{-\infty}^{\infty} e^{-\sqrt{x}\,s^2}\, ds = \frac{1}{2\pi}\,\frac{\sqrt{\pi}}{x^{1/4}}\, e^{-\frac{2}{3}x^{3/2}}$$

$$= \frac{1}{2\sqrt{\pi}}\,\frac{1}{x^{1/4}}\, e^{-\frac{2}{3}x^{3/2}} \quad \text{for } x \to +\infty$$

$\boxed{X \ll 0}$ Harder! Now $t = \pm i\sqrt{|x|}$ and we need to account for both! As $x<0$, we have

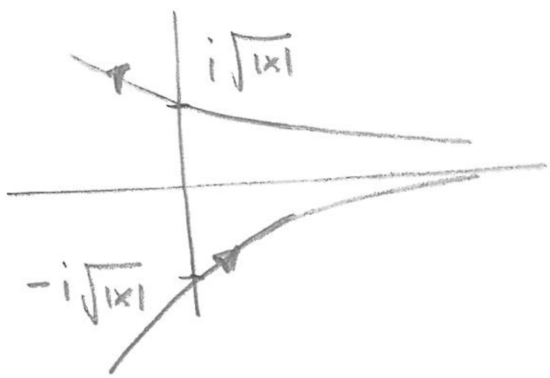
$\varphi(t) = -|x|t - \frac{1}{3}t^3$, still $\varphi''(t) = -2t \Rightarrow$

$$\varphi(i\sqrt{|x|} + \xi) \approx -\frac{2}{3} i |x|^{3/2} - i\sqrt{|x|}\,\xi^2$$

Now: Choose the contour such that the real part has a max and the phase is constant

$$\xi = e^{3\pi i/4} s \Rightarrow \xi^2 = e^{3\pi i/2} s^2 = -is^2$$

$$\Rightarrow \varphi(i\sqrt{|x|} + \xi) \approx -\frac{2}{3} i |x|^{3/2} - \sqrt{|x|}\, s^2$$



$\Rightarrow$ contribution is

$$\frac{1}{2\pi i} e^{-\frac{2}{3} i|x|^{3/2}} e^{3\pi i/4} \frac{\sqrt{\pi}}{|x|^{1/4}}$$

$$= \frac{-1}{2i\sqrt{\pi}} e^{-i\pi/4} \frac{1}{|x|^{1/4}} e^{-i\frac{2}{3}|x|^{3/2}}$$

similar for $t = -i\sqrt{|x|}$: contribution is then

$$\frac{1}{2i\sqrt{\pi}} e^{i\pi/4} \frac{1}{|x|^{1/4}} e^{i\frac{2}{3}|x|^{3/2}} \Rightarrow \text{combined}$$

$$Ai(x) \approx \frac{1}{\sqrt{\pi}} \frac{1}{|x|^{1/4}} \sin\left(\frac{2}{3}|x|^{3/2} + \frac{\pi}{4}\right)$$