Math Methods

Lecture 10

Method of stationary phase

Consider ($\lambda \gg 1$) $\quad I(\lambda) = \int_a^b f(x)\, e^{i\lambda\phi(x)}\, dx$

Assume that $\phi'(x_s) = 0$. The major contribution to $I(\lambda)$ comes from a small neighborhood around x_s: Write

$$I(\lambda) = e^{i\lambda\phi(x_s)} \int_a^b f(x)\, \underbrace{e^{i\lambda(\phi(x)-\phi(x_s))}}_{\text{highly oscillatory}}\, dx$$

$$\phi(x) = \phi(x_s) + \frac{1}{2}\phi''(x_s)(x-x_s)^2 + \cdots \quad , \quad \xi = x - x_s$$

$$I(\lambda) \approx e^{i\lambda\phi(x_s)} \int_{-\infty}^{\infty} f(x_s)\, e^{i\frac{\lambda}{2}\phi''(x_s)\xi^2}\, d\xi \qquad (\phi''(x_s) > 0)$$

$$\boxed{I(\lambda) \approx f(x_s)\, e^{i\lambda\phi(x_s)} \sqrt{\frac{2\pi i}{\lambda\phi''(x_s)}}}$$

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Example :

Bessel function: $J_n(\lambda) = \int_0^1 \cos(n\pi x - \lambda \sin \pi x)\, dx$

$$= Re\left(\int_0^1 \underset{\uparrow\atop f(\lambda)}{e^{n\pi i x}}\ \underset{\uparrow\atop e^{i\lambda\phi(x)}}{e^{-i\lambda \sin \pi x}}\, dx \right)$$

$\phi(x) = -\sin(\pi x)$

and $\phi'(x) = -\pi\cos(\pi x)$ is 0 at $x_s = \frac{1}{2}$

$\phi(x_s) = -1 \quad \phi''(x_s) = \pi^2 \sin(\pi x_s) = \pi^2$

$f(x_s) = e^{n\pi i/2}$

$$\Rightarrow J_n(\lambda) \approx Re\left(e^{n\pi i/2} e^{-i\lambda} \sqrt{\frac{2\pi i}{\lambda \pi^2}} \right)$$

Now $\sqrt{\dfrac{2\pi i}{\lambda\pi^2}} = \sqrt{\dfrac{2}{\lambda\pi}}\, e^{i\pi/4}$

$$J_n(\lambda) \approx \sqrt{\frac{2}{\lambda\pi}}\, Re\left(e^{-i(\lambda - \frac{n\pi}{2} - \frac{\pi}{4})} \right)$$

$$\boxed{J_n(\lambda) \approx \sqrt{\frac{2}{\lambda\pi}} \cos\left(\lambda - \frac{n\pi}{2} - \frac{\pi}{4}\right)}$$

Matlab:

```
z = linspace (2, 30, 1000);
                                   n
y = sqrt (2./(z*pi)).* cos(z - 2*pi/2 - pi/4);
plot (z, besselj(2,z), z, y)
```

Method of steepest descent

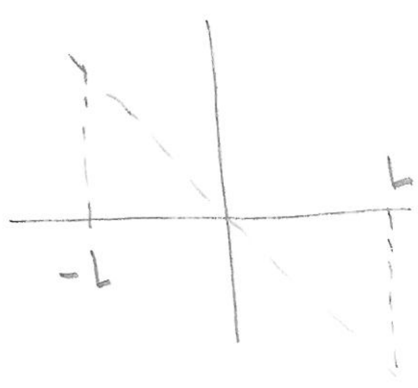
How can we evaluate $\int_{-\infty}^{\infty} e^{iax^2} dx$?

We know $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$. We will show that

$$J = \int_{-\infty}^{\infty} e^{-iax^2} dx = \sqrt{\frac{\pi}{ia}} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{i\pi}{4}}$$

$$e^{-iax^2} = \cos(ax^2) - i\sin(ax^2)$$

most contributions will come from region around zero



$$e^{-iaz^2} \begin{cases} \text{along } & z = L - iw \quad (w>0) \\ \text{and } & z = -L + iw \quad (w>0) \\ \text{small} & \end{cases}$$

$$(L-iw)^2 = L^2 - w^2 - 2iwL$$

$$-iaz^2 = -ia(L^2 - w^2) - 2awL \leftarrow \text{decaying}$$

Therefore: we can integrate along this line

$$z = (1-i)\,t \qquad 1-i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$$

$$dz = (1-i)\,dt \qquad z^2 = -2i t^2$$

$$\Rightarrow \quad \mathcal{J} = \left(\int_{-\infty}^{\infty} e^{-2at^2}\,dt\right)\cdot \sqrt{2}\, e^{-i\frac{\pi}{4}} = \sqrt{\frac{\pi}{2a}}\,\sqrt{2}\, e^{-i\frac{\pi}{4}}$$

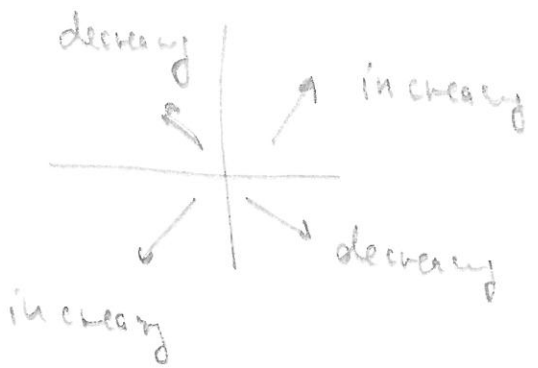
$$\Rightarrow \quad \mathcal{J} = \sqrt{\frac{\pi}{a}}\, e^{-i\frac{\pi}{4}} \quad \text{as claimed}$$

Idea: distort integrand in a way that it decays as fast as possible away from the stationary point.

Consider e^{-iaz^2} and $z = x+iy$

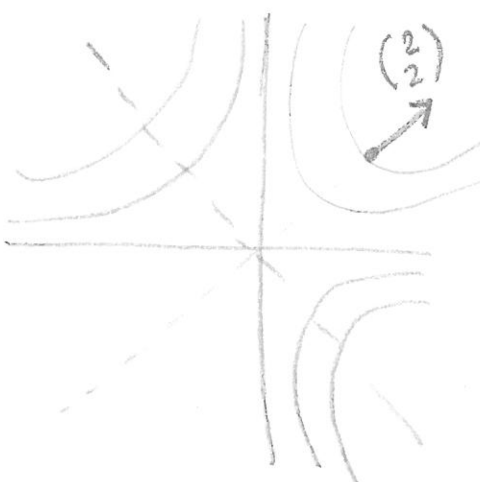
$$\Rightarrow -iaz^2 = -ia(x^2-y^2+2ixy) = -ia(x^2-y^2) + 2xy$$

$\Rightarrow$



Steepest descent lines:
- perpendicular to lines of const. $\mathrm{Re}\, f(z)$
- phase is const.

$\Rightarrow$ all contributions are coherent and add up.



<u>General:</u> write $e^{\lambda u(x,y)} \; e^{i\lambda v(x,y)}$

For a point $z_0 = (x_0, y_0)$, we know that $\nabla u \big|_{z_0}$ defines the direction of the steepest ascent, $-\nabla u \big|_{z_0}$ the direction of the steepest descent.

$z = x + iy \Rightarrow -iaz^2 = 2xy - ia(x^2 - y^2)$

$\Rightarrow \nabla u = 2\begin{pmatrix} y \\ x \end{pmatrix}$ e.g. $\nabla u \big|_{(1,1)} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

General saddle point of order $N-1$:

$$p'(z_0), \dots, p^{(N-1)}(z_0) = 0 \,, \quad p^{(N)}(z_0) \neq 0$$

$$\Rightarrow \quad p(z) = p(z_0) + \frac{(z-z_0)^N}{N!} p^{(N)}(z_0) + \cdots$$

$$z = z_0 + \rho e^{i\theta} \,, \quad p^{(N)}(z_0) = a e^{i\alpha}$$

$$\Rightarrow \quad p(z) - p(z_0) \sim \frac{\rho^N a e^{i(N\theta + \alpha)}}{N!}$$

Curves of steepest ascent / descent are given by $\mathrm{Im}\,(p(z) - p(z_0)) = 0 \Rightarrow \sin(N\theta + \alpha) = 0$

hence $N\theta + \alpha = k\pi$, $k \in \mathbb{Z}$

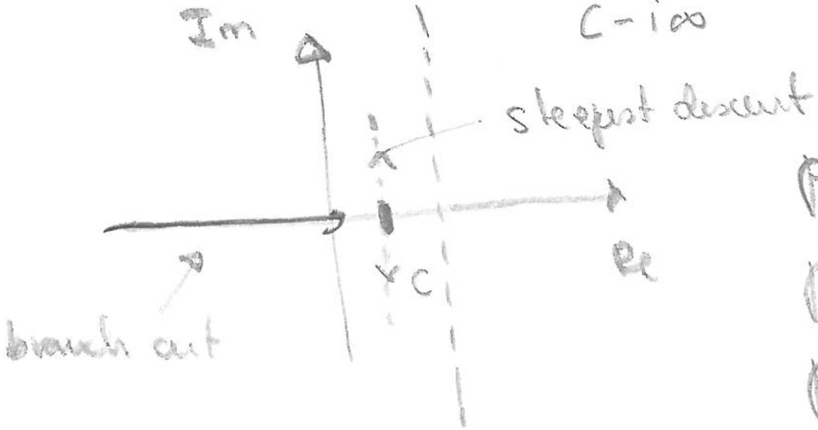
$$\Rightarrow \quad p(z) - p(z_0) = u(x,y) - u(x_0, y_0) \sim \frac{\rho^N a}{N!} \cos(N\theta + \alpha)$$

and we want this to be <u>negative</u>.

$$\Rightarrow \theta = -\frac{\alpha}{N} + (2k+1)\frac{\pi}{N} \Rightarrow \cos(N\theta + \alpha) = -1.$$

This defines the curves of steepest descent.

Ex. $$I(\lambda) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{e^{\lambda(at - \sqrt{t})}}{t}\, dt \qquad a, c > 0$$



$$p(t) = at - \sqrt{t}$$

$$p'(t) = a - \frac{1}{2}t^{-1/2} = a - \frac{1}{2\sqrt{t}}$$

$$p''(t) = \frac{1}{4}t^{-3/2}$$

Saddle. $p'(t) = 0 \Rightarrow t = t_0 = \frac{1}{4a^2}$, $p''(t_0) = 2a^3$

Here, $N=2$, $p''(t_0) = 2a^3$ does not have a phase,

here directions of steepest descent are

$$\theta = \frac{\pi}{2}, -\frac{\pi}{2}$$

$\Rightarrow$ Basic approximation straight line in direction of steepest descent

$$t = \frac{1}{4a^2} + T\,i \quad \text{small}$$

, we know derivative vanishes...

$$\Rightarrow at - \sqrt{t} = \underbrace{\frac{a}{4a^2} - \sqrt{\frac{1}{4a^2}}}_{(T=0)} - \frac{1}{2}\,\underbrace{2a^3}_{\text{second derivative}}\,T^2$$

$$= -\frac{1}{4a} - a^3 T^2 \quad \text{and} \quad \frac{1}{t} \approx 4a^2$$

$$\Rightarrow I(\lambda) = \frac{1}{2\pi i}\int \frac{e^{\lambda(at-\sqrt{t})}}{t}\,dt$$

$$\approx \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{\lambda}{4a}}\, e^{-a^3T^2\lambda}\, 4a^2 i\, dT$$

$$= e^{-\frac{\lambda}{4a}}\,\frac{2a^2}{\pi}\int_{-\infty}^{\infty} e^{-a^3T^2\lambda}\,dT = e^{-\frac{\lambda}{4a}}\,\frac{2a^2}{\pi}\sqrt{\frac{\pi}{a^3\lambda}}$$

$$= 2\sqrt{\frac{a}{\pi\lambda}}\, e^{-\lambda/4a}$$