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Laplace Transforms

$$L\left[f(x)\right] = \int_0^\infty e^{-sx} f(x)\,dx = F(s)$$

Most important property:

$$L(f'(x)) = \int_0^\infty e^{-sx} f'(x)\,dx = e^{-sx} f(x)\Big|_0^\infty$$

$$+ \int_0^\infty s e^{-sx} f(x)\,dx = sF(s) - f(0)$$

$$\left(\text{if } \lim_{x\to\infty} e^{-sx} f(x) = 0\right)$$

Similar to Fourier transform:

differential operator $\longrightarrow$ multiplication.

But : Here $f(0)$ is important.

Laplace transform is good for initial value problems.

Difficult part: inverse transform.

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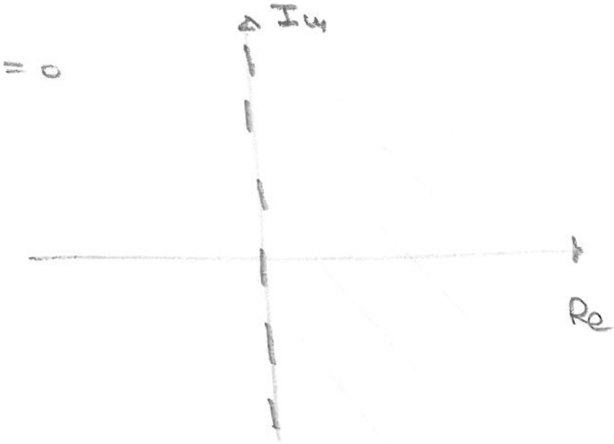
Usually: $s = \sigma + i\omega \in \mathbb{C}$. One can show that there is a σ_c such that $F(s)$ is analytic in the region of $\mathrm{Re}(s) > \sigma_c$.

Example: $f(x) = 1 \Rightarrow F(s) = \int_0^\infty e^{-sx}\,dx = \frac{1}{s} - \lim_{x \to \infty} e^{-sx}$

Clearly $\lim_{x\to\infty} e^{-sx} = 0$ for $s > 0$, but divergence for $s < 0$ $\Rightarrow \sigma_c = 0$



$\sigma_c = 0$

More examples:

$f(x) = x^n$

$$\int_0^\infty x^n e^{-sx}\,dx$$

$$= \frac{-x^n e^{-sx}}{s}\Big|_0^\infty + \frac{n}{s}\int_0^\infty x^{n-1} e^{-sx}\,dx \text{ , iterate}$$

$$\Rightarrow \boxed{L(f) = \frac{n!}{s^{n+1}}} \quad (\sigma > 0)$$

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$$f(x) = e^{ax} \Rightarrow L(f) = \int_0^\infty e^{-sx} e^{ax}\,dx = \frac{1}{s-a} \qquad \sigma > a$$

$$f(x) = \sin ax \Rightarrow \int_0^\infty e^{-sx} \sin ax\,dx = F(s)$$

Integrate by parts twice, be careful about boundaries

$$F(s) = \int_0^\infty \underset{u}{e^{-sx}} \underset{v'}{\sin ax}\,dx = -\frac{e^{-sx}\cos ax}{a}\Big|_0^\infty - \frac{1}{a}\int_0^\infty \underset{u}{s e^{-sx}} \underset{v'}{\cos ax}\,dx$$

$$= \frac{1}{a} - \frac{s}{a^2}\left\{\int_0^\infty s e^{-sx} \sin ax\,dx\right\} = \frac{1}{a} - \frac{s^2}{a^2} F(s)$$

$$F(s)\left(1 + \frac{s^2}{a^2}\right) = \frac{1}{a} \Rightarrow F(s) = \frac{1}{a\left(1+\frac{s^2}{a^2}\right)} = \frac{a}{s^2+a^2} \qquad (\sigma > 0)$$

Note: $f(x) = \cos ax \Rightarrow \quad F(s) = \dfrac{s}{s^2+a^2} \qquad (\sigma > 0)$

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Shifting theorems:

(1) $\mathcal{F}(s) = L(f(x)) \Rightarrow \mathcal{F}(s+a) = L(e^{-ax} f(x))$

$\sigma > \sigma_c - \mathrm{Re}(a)$

(Clearly $\int_0^\infty e^{-ax} e^{-sx} f(x)\,dx = \int_0^\infty e^{-(s+a)x} f(x)\,dx$

$= \mathcal{F}(s+a)$

(converges for $\mathrm{Re}(s+a) > \sigma_c$)

Example: $L(e^{-ax} \sin bx) = \dfrac{b}{(s+a)^2 + b^2}$

(2) Consider $\theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ $\quad (\theta'(x) = \delta(x))$

Then $e^{-sx_0} \mathcal{F}(s) = L(f(x-x_0)\,\theta(x-x_0))$

$$\int_0^\infty f(x-x_0)\,\theta(x-x_0)\, e^{-sx}\,dx = \int_{x_0}^\infty f(x-x_0)\, e^{-sx}\,dx \qquad \tilde{x} = x - x_0$$

$$= \int_0^\infty f(\tilde{x})\, e^{-s(\tilde{x}+x_0)}\,d\tilde{x} = e^{-sx_0}\,\mathcal{F}(s)$$

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Higher derivatives: $L(f'') = s^2 F(s) - s f(0) - f'(0)$

$$L\left(f^{(n)}(x)\right) = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$$

If $g(x) = \int_0^x f(u)\,du$, $\lim_{x\to\infty} e^{-sx} g(x) = 0$

$$\Rightarrow \quad L(g(x)) = \frac{F(s)}{s}$$

(Clearly, $L(g'(x)) = s\,L(g(x)) - g(0)$

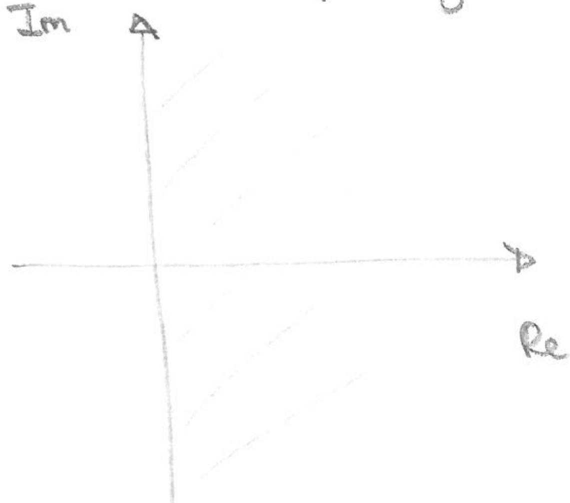
$F(s) \quad = s\,L(g(x))$) $\square$

Multivalued functions: $f(x) = \frac{1}{\sqrt{x}}$

$$\int_0^\infty \frac{e^{-sx}}{\sqrt{x}}\,dx = ?$$

$$sx = u^2 \Rightarrow u = \sqrt{s}\sqrt{x}, \quad du = \frac{\sqrt{s}}{2\sqrt{x}}\,dx$$

$$\Rightarrow \int_0^\infty \frac{e^{-sx}}{\sqrt{x}} = \frac{2}{\sqrt{s}} \int_0^\infty e^{-u^2}\,du = \sqrt{\frac{\pi}{s}}$$



This is true for real s.
$\Rightarrow$ We take in the complex plane $\mathrm{Re}(s) > 0$.

Two-sided Laplace transform

$$\mathcal{F}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-sx}\, dx$$

$$\mathcal{F}(\sigma + i\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-\sigma x}\, e^{-i\omega x}\, dx$$ related

to Fourier transform $\Rightarrow$

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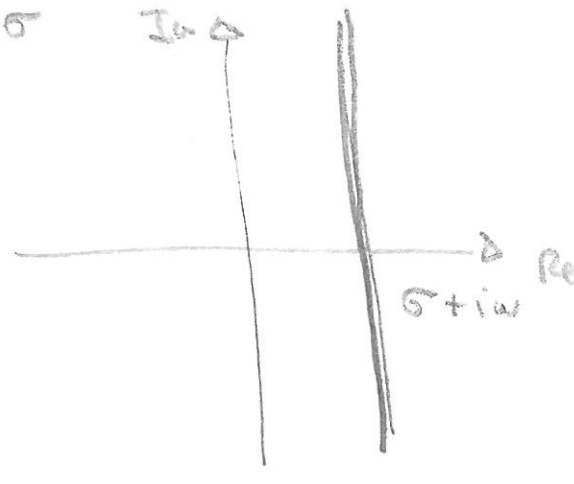
$$f(x)\,e^{-\sigma x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma+i\omega)\,e^{i\omega x}\,d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma+i\omega)\,e^{\sigma x}\,e^{i\omega x}\,d\omega$$

$s = \sigma + i\omega$

integral along a vertical line

$\mathrm{Re}(s) = \sigma$



$ds = i\,d\omega$

$$\Rightarrow\quad f(x) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} F(s)\,e^{sx}\,ds$$

However, we need to make sure that σ is in the region where $F(s)$ converges.

Clearly, if $\mathcal{L}(f)$ is the two-sided Laplace transform and $L(f)$ is the one-sided Laplace transform $F(s) = L(f) = \mathcal{L}(f\,\theta(x))$

$$\Rightarrow f(x)\,\theta(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\, e^{sx}\, ds$$

σ needs to be to the right of all singularities

<u>Summary:</u> $F(s) = \int_0^\infty e^{-sx} f(x)\,dx$

is analytic for $\mathrm{Re}(s) > \sigma_c$. Then $f(x)$, $x > 0$ is given by

$$f(x) = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\sigma - i\omega}^{\sigma + i\omega} e^{sx} F(s)\, ds$$

(σ is arbitrary, but $\sigma > \sigma_c$).

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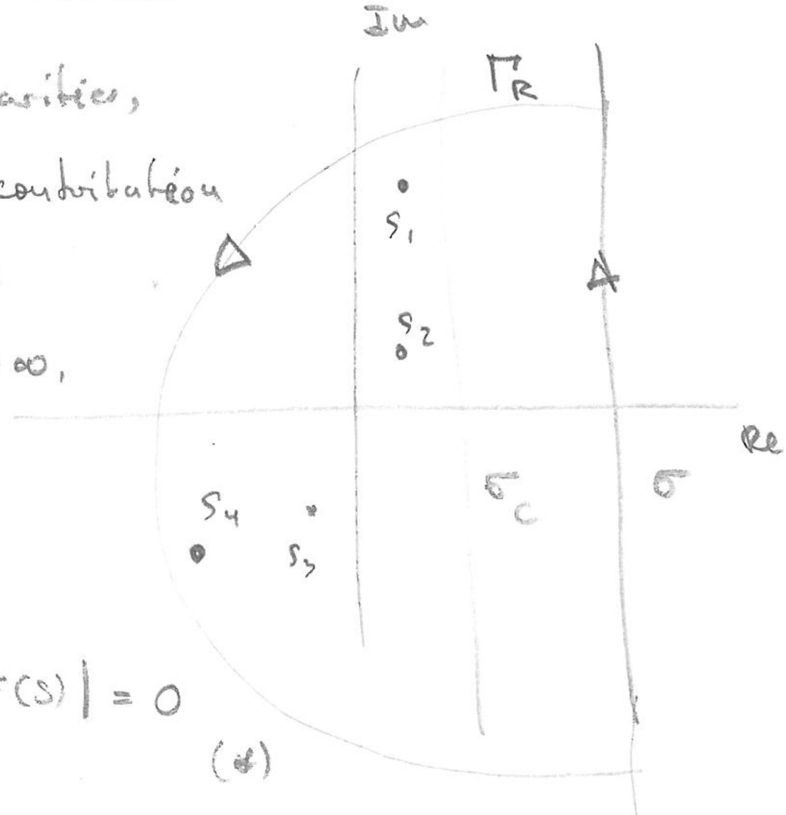
Often: Residue theorem

include all singularities, make sure that contribution along semi-circle goes to 0 for $R \to \infty$.

One sufficient condition:

$$\lim_{R\to\infty} \max_{s \in \Gamma_R} |F(s)| = 0 \qquad (*)$$



<u>Example:</u> $F(s) = \dfrac{1}{s^2-3s+2}$, poles at $s_1 = 1$, $s_2 = 2$

Choose $\sigma = 3$. Clearly as $|s-3| \to \infty$, we have $|s-1|, |s-2| \to \infty$

$\Rightarrow$ $(*)$ is satisfied

$$f(x) = \mathrm{Res}\left(\frac{e^{sx}}{(s-1)(s-2)};\ s=1\right) + \mathrm{Res}\left(\frac{e^{sx}}{(s-1)(s-1)};\ s=2\right)$$

$$\Rightarrow \quad f(x) = \frac{e^x}{1-2} + \frac{e^{2x}}{2-1} = -e^x + e^{2x}.$$

Example: Solution of an ODE

$$f''(x) + f(x) = 1 \qquad f(0) = f'(0) = 0$$

$$s^2 F(s) + F(s) = \frac{1}{s}$$

$$\left(L(f'') = s^2 F(s) - s f(0) - f'(0) \right)$$

$$\Rightarrow \quad F(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\Rightarrow \quad f(x) = 1 - \cos x.$$