# Math Methods

## Lecture 8

### Fourier series:

Trigonometric series: $\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$

Fourier series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Dirichlet theorem: If $F$ is periodic with period $2\pi$ and if $F'$ is continuous or has (at most) a finite number of discontinuities in $[0, 2\pi]$, then its Fourier series converges to

1. $f(x)$ if $x$ is a point of continuity
2. $\frac{f(x+0) + f(x-0)}{2}$ if $x$ is a point of discontinuity.

Note: $L = \frac{\partial^2}{\partial x^2}$ with periodic b.c. is self-adjoint.

Over $2\pi$: eigenfunctions are $g_n(x) = \sin nx$

and $h_n(x) = \cos nx$

and $\tilde{h}(x) = c = \text{const}$.

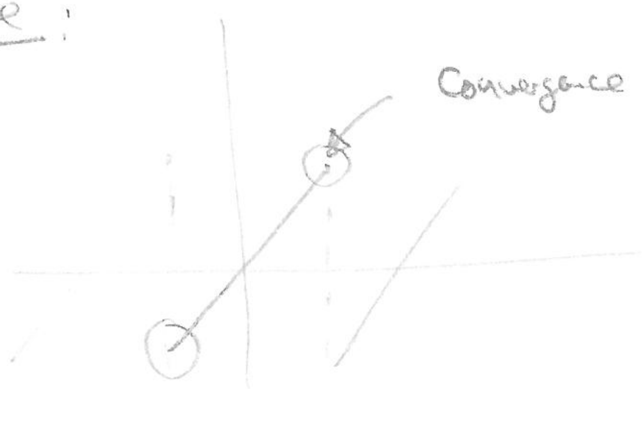
Note: symmetries simplify the coefficients.

Note: More general: (period $\lambda$)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inkx}$$

$$c_n = \frac{1}{\lambda} \int_0^{\lambda} f(x)\, e^{-inkx}\, dx \qquad \left(k = \frac{2\pi}{\lambda}\right)$$

Convergence:

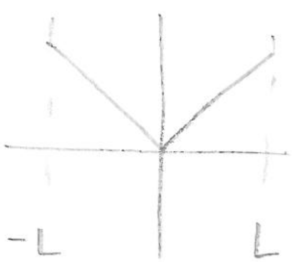


Convergence usually not good

(Gibbs phenomenon)

**Example:** (a) $f(x) = \begin{cases} -x & \text{for } -L \le x < 0 \\ x & \text{for } 0 \le x \le L \end{cases}$ $\left(k = \frac{2\pi}{2L} \text{ or } Lk = \pi\right)$

Even $\Rightarrow$ only cos terms and constant



$$a_0 = \frac{2}{L}\int_0^L x\,dx = \frac{2}{L}\frac{L^2}{2} = L$$

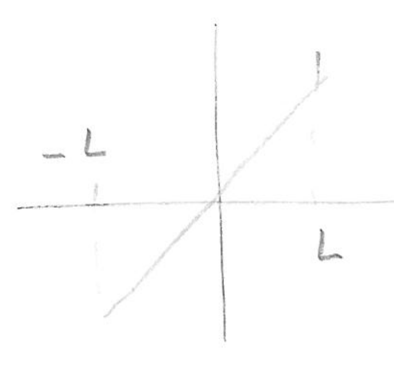
$$a_n = \frac{2}{L}\int_0^L f(x)\cos(nkx)\,dx = \cdot$$

$$[\cdots] = \frac{2L}{\pi^2}\frac{(-1)^n - 1}{n^2} = \begin{cases} \frac{-4L}{n^2\pi^2} & n = 1, 3, \ldots \\ 0 & n = 2, 4, \ldots \end{cases}$$

quadratic in n $\Rightarrow$ fast convergence

$$\Rightarrow f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}$$

(b) $f(x) = x$ for $-L \le x \le L$



$$b_n(x) = \frac{2}{L}\int_0^L f(x)\sin(nkx)\,dx$$

$$= \frac{2L}{\pi}\frac{(-1)^{n+1}}{n}$$

linear in n $\Rightarrow$ slower convergence

<u>Fourier transform</u>:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \qquad k_n = \frac{2\pi}{\lambda} n$$

$$c_n = \frac{1}{\lambda} \int_{-\frac{\lambda}{2}}^{\lambda/2} f(x)\, e^{-ik_n x} dx$$

$\Delta k = \frac{2\pi}{\lambda} \Delta n$ , $(\Delta n = 1)$ hence $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \Delta n$

$$\Rightarrow \quad f(x) = \sum_{n=-\infty}^{\infty} C_\lambda(k_n)\, e^{ik_n x} \Delta k$$

$C_\lambda(k_n) = \frac{\lambda}{2\pi} c_n$ . Now we can look at the limit $\lambda \to \infty$ $(\Delta k \to 0)$ $\Rightarrow$ $C_\lambda(k_n) \to C(k)$

(continuous)

$$c(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx$$

$$f(x) = \int_{-\infty}^{\infty} c(k)\, e^{ikx} dk$$

$$F(k) = \sqrt{2\pi}\, c(k)$$

Fourier transform:
$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx$$
$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\,e^{ikx}\,dk$$

Convolution Theorem:
$$(f*g)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du$$

Theorem: $F=\mathcal{F}(f)$, $G=\mathcal{F}(g)$
$$\Rightarrow \quad F(k)\,G(k)=\mathcal{F}(f*g)$$

Proof:
$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx$$
$$G(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\,e^{-ikx}\,dx$$

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$$F(k)\,G(k) = \frac{1}{2\pi}\iint f(x)\,g(x')\,e^{-ik\overbrace{(x+x')}^{u = x+x'}}\,dx\,dx'$$

$$= \frac{1}{2\pi}\iint f(x)\,g(u-x)\,e^{-iku}\,dx\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int e^{-iku}\left\{\frac{1}{\sqrt{2\pi}}\int f(x)\,g(u-x)\,du\right\}dx$$

$$= \mathcal{F}(f * g)$$

Note: $$a(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^*(x)\,f(x+z)\,dx$$

is called autocorrelation function.

It is easy to show that

$$a(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |F(k)|^2\,e^{ikx}\,dk$$

so-called power spectrum of $f$

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## Fast - Fourier - Transform (FFT)

Discrete: $f_0, f_1, \dots f_{N-1}$, sample points: $x_j = j \cdot \Delta x$

Frequencies: $k_n = \dfrac{2\pi n}{N \cdot \Delta x}$ $\quad n = 0, 1, 2, \dots, N-1$

or $n = -\frac{N}{2}+1, -\frac{N}{2}+2, \dots, 0, \dots \frac{N}{2}$

(periodicity)

Fourier transform (sign as in "Numerical Recipes")

$$F(k_n) = \int_{-\infty}^{\infty} f(x)\, e^{ik_n x}\, dx \approx \sum_{j=0}^{N-1} f_j\, e^{ik_n x_j} (\Delta x)$$

$$= \sum_{j=0}^{N-1} f_j\, e^{2\pi i\, jn/N} (\Delta x) = (\Delta x) \underbrace{\sum_{j=0}^{N-1} f_j\, e^{2\pi i\, jn/N}}_{F_n}$$

inverse: $$f_j = \frac{1}{N} \sum_{n=0}^{N-1} F_n\, e^{-2\pi i\, jn/N}$$

$$\left( \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i\, j'n/N} \left( \sum_{j=0} f_j\, e^{2\pi i\, jn/N} \right) \right.$$

$$\left. = \sum_{j=0}^{N-1} f_j \left( \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i (j-j')n/N} \right) = \sum_{j=0} f_j\, \delta_{jj'} = f_{j'} \right)$$

**Note:** We can write $F_n = \sum_{k=0}^{N-1} W^{nk} f_k$

$$W = e^{2\pi i/N}$$

$$F_n = \sum_{k=0}^{N-1} e^{2\pi i k n/N} f_k$$

$$= \sum_{k=0}^{N/2-1} e^{2\pi i n(2k)/N} f_{2k} + \sum_{k=0}^{N/2-1} e^{2\pi i n(2k+1)/N} f_{2k+1}$$

$$= \sum_{k=0}^{N/2-1} e^{2\pi i n k/(N/2)} f_{2k} + W^n \sum_{k=0}^{N/2-1} e^{2\pi i n k/(N/2)} f_{2k+1}$$

$$= F_n^e + W^n F_n^o$$

even (length: $N/2$)

odd (length: $N/2$)

Example:

$\boxed{F_0 \; F_1 \; F_2 \; F_3 \; F_4 \; F_5 \; F_6 \; F_7}$

$\boxed{F_0^e \; F_2^e \; F_4^e \; F_6^e}$ $\boxed{F_1^o \; F_3^o \; F_5^o \; F_7^o}$

$\boxed{F_0^{ee} \; F_4^{ee}}$ $\boxed{F_2^{eo} \; F_6^{eo}}$ $\boxed{F_1^{oe} \; F_5^{oe}}$ $\boxed{F_3^{oo} \; F_7^{oo}}$

Ordering: bit reversal

| | | | |
|---|---|---|---|
| 000 | 0 | 0 | |
| 001 | 1 | 4 | (100) |
| 010 | 2 | 2 | |
| 011 | 3 | 6 | (110) |
| 100 | 4 | 1 | (001) |
| 101 | 5 | 5 | |
| 110 | 6 | 3 | (011) |
| 111 | 7 | 7 | |

Typical structure of a "divide and conquer" algorithm.
↳ much faster than direct matrix multiplication

FFT can be generalized to higher dimensions.

FFT in practice:


$x = \text{linspace}(x_{min}, x_{max} - \frac{x_D}{N}, N)$ $\quad x_D = x_{max} - x_{min}$

$\text{dom} = \frac{2\pi}{x_D}$ , $\text{om} = ([0:N/2, -N/2+1:-1]) * \text{dom}$

↳ see example (fftTest.m)

Use of Fourier series / transform to solve linear PDEs:

Assume $u_t = Lu$ and $L = L^+$ with an ONB $\{\psi_k\}$ and $L\psi_k = \lambda_k \psi_k$. Then expand $u(x,t) = \sum_k \alpha_k(t)\psi_k(x)$

$$\Rightarrow \quad u_t = \sum_k \dot{\alpha}_k \psi_k = Lu = L\sum_k \alpha_k \psi_k = \sum_k \alpha_k \lambda_k \psi_k$$

or $\dot{\alpha}_k = \lambda_k \alpha_k$ which are (decoupled) ODEs.

$$\Rightarrow \quad \alpha_k(t) = e^{\lambda_k t}\alpha_k(0) \quad \text{and} \quad u(x,t) = \sum_k e^{\lambda_k t}\alpha_k(0)\psi_k(x)$$

is the solution of the PDE.

The initial conditions $\alpha_k(0)$ are found by projection $u(x,t=0) = u_0(x) = \sum_k \alpha_k(0)\psi_k$

$$\Rightarrow \quad \alpha_k(0) = (\psi_k(x), u_0(x)) \quad \text{since } \{\psi_k\}$$

is an ONB.

If $L$ has a continuous spectrum: sums become integrals (see ex using Fourier transform)

**Example:** Use of Fourier transform to solve heat equation:

$$(*) \qquad u_t = \varkappa u_{xx}, \qquad u(t=0,x) = e^{-x^2/2}$$

We use the Fourier transform to diagonalize $L = \varkappa \partial_x^2$.

$$\begin{tikzcd}
u_0 \arrow[r, "\text{Fourier}"] \arrow[d, "\substack{\text{solution}\\ \text{of}\\ \text{PDE}}"'] & \hat{u}_0(k) \arrow[d, "\substack{\text{simple}\\ \text{ODEs}}"] \\
u(t,x) & \hat{u}_0(t,k) \arrow[l, "\substack{\text{inverse}\\ \text{Fourier}}"]
\end{tikzcd}$$

$$\hat{u}(t,k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t,x)\, e^{-ikx}\, dx$$

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(t,k)\, e^{ikx}\, dk \quad \Rightarrow \quad \boxed{\partial_x \to ik}$$

Therefore: $(*)$ in Fourier domain becomes

$$\hat{u}_t = -\varkappa k^2 \hat{u}$$

<u>Here:</u> $$\hat{u}(t=0,k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2 + 2ikx + (ik)^2 - (ik)^2\right)}\, dx = e^{-\frac{1}{2}k^2}$$

solution of $\hat{u}_t = -\varkappa k^2 \hat{u}$ is $\hat{u}(t,k) = e^{-\varkappa k^2 t}\, \hat{u}(0,k)$

$$\Rightarrow \quad u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\varkappa k^2 t}\, e^{-\frac{1}{2}k^2}\, e^{ikx}\, dk$$

$$\underset{\text{(some algebra)}}{=} \frac{1}{\sqrt{2\varkappa t + 1}}\, e^{-x^2 / 2(2\varkappa t + 1)}$$