

Fourier Series:

Trigonometric series: $\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$

Fourier series: $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Dirichlet theorem: If f is periodic with period 2π and if f' is continuous or has (at most) a finite number of discontinuities in $[0, 2\pi]$, then its Fourier series converges to

1. $f(x)$ if x is a point of continuity
2. $\frac{f(x+0) + f(x-0)}{2}$ if x is a point of discontinuity.

Note: $L = \frac{\partial^2}{\partial x^2}$ with periodic b.c. is self-adjoint.

Over 2π : eigenfunctions are $g_n(x) = \sin nx$
and $h_n(x) = \cos nx$
and $\tilde{h}(x) = c = \text{const.}$

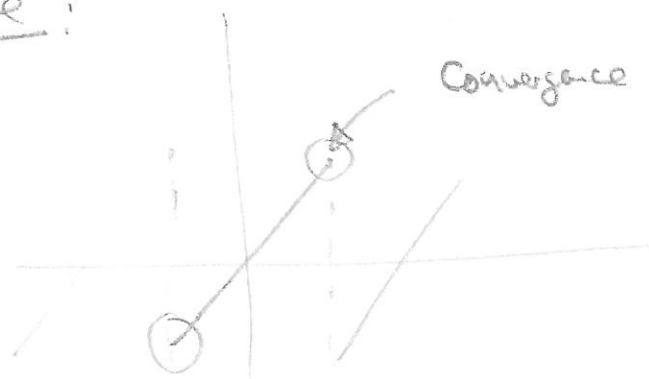
Note: Symmetries simplify the coefficients.

Note: More general: (period λ)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inkx}$$

$$c_n = \frac{1}{\lambda} \int_0^{\lambda} f(x) e^{-inkx} dx \quad \left(k = \frac{2\pi}{\lambda}\right)$$

Convergence:



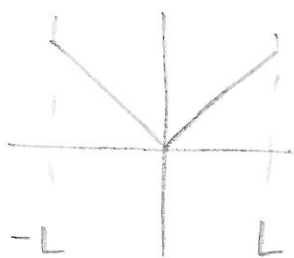
Convergence usually not good
(Gibbs phenomenon)

Example: (a) $f(x) = \begin{cases} -x & \text{for } -L \leq x < 0 \\ x & \text{for } 0 \leq x \leq L \end{cases}$

$$\left(k = \frac{2\pi}{2L} \right)$$

$$\text{or } Lk = \pi$$

Even $\Rightarrow$ only cos terms and constant



$$a_0 = \frac{2}{L} \int_0^L x \, dx = \frac{2}{L} \frac{L^2}{2} = L$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos(nkx) \, dx = \dots$$

quadratic in n
 $\Rightarrow$ fast convergence

$$[\dots] = \frac{2L}{\pi^2} \frac{(-1)^n - 1}{n^2} = \begin{cases} \frac{-4L}{n^2 \pi^2} & n = 1, 3, \dots \\ 0 & n = 2, 4, \dots \end{cases}$$

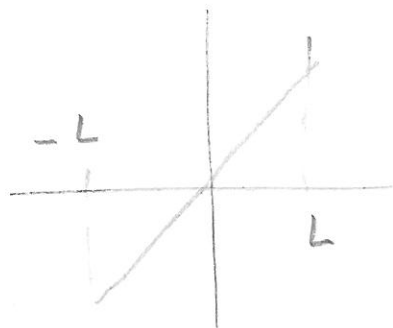
$$\Rightarrow f(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left(\frac{(2n-1)\pi x}{L}\right)$$

(b) $f(x) = x$ for $-L \leq x \leq L$

$$b_n(x) = \frac{2}{L} \int_0^L f(x) \sin(nkx) \, dx$$

$$= \frac{2L}{\pi} \frac{(-1)^{n+1}}{n}$$

linear in $n \rightarrow$ slow convergence



Fourier transform:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x}$$

$$k_n = \frac{2\pi}{\lambda} n$$

$$c_n = \frac{1}{\lambda} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(x) e^{-ik_n x} dx$$

$$\Delta k = \frac{2\pi}{\lambda} \Delta n$$

$$(\Delta n = 1)$$

$$\text{hence } f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \Delta n$$

$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty} C_\lambda(k_n) e^{ik_n x} \Delta k$$

$$C_\lambda(k_n) = \frac{\lambda}{2\pi} c_n. \text{ Now we can look at the}$$

$$\text{limit } \lambda \rightarrow \infty (\Delta k \rightarrow 0) \Rightarrow C_\lambda(k_n) \rightarrow C(k)$$

$$C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \quad (\text{continuous})$$

$$f(x) = \int_{-\infty}^{\infty} C(k) e^{ikx} dk$$

$$F(k) = \sqrt{2\pi} C(k)$$

Fourier transform:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

Convolution Theorem:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) g(x-u) du$$

Theorem:

$$F = \mathcal{F}(f), \quad G = \mathcal{F}(g)$$

$$\Rightarrow F(k) G(k) = \mathcal{F}(f * g)$$

Proof:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ikx} dx$$

$$F(u) G(u) = \frac{1}{2\pi} \iint f(x) g(x') e^{-ik(x+x')} dx dx'$$

$u = x+x'$

$$= \frac{1}{2\pi} \iint f(x) g(u-x) e^{-iku} dx du$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{-iku} \left\{ \frac{1}{\sqrt{2\pi}} \int f(x) g(u-x) dx \right\} du$$

$$= F(g)$$

Note:

$$a(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^*(x) f(x+z) dx$$

is called autocorrelation function.

It is easy to show that

$$a(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |F(k)|^2 e^{ikz} dk$$

↑
so-called power spectrum of f

Fast - Fourier - Transform (FFT)

Discrete: $f_0, f_1, \dots, f_{N-1}$, sample points: $x_j = j \cdot \Delta x$

Frequencies: $k_n = \frac{2\pi n}{N \cdot \Delta x}$ $n = 0, 1, 2, \dots, N-1$

or $n = -\frac{N}{2} + 1, -\frac{N}{2} + 2, \dots, 0, \dots, \frac{N}{2}$
(periodicity)

Fourier transform (sign as in "Numerical Recipes")

$$\begin{aligned} F(k_n) &= \int_{-\infty}^{\infty} f(x) e^{ik_n x} dx \approx \sum_{j=0}^{N-1} f_j e^{ik_n x_j (\Delta x)} \\ &= \sum_{j=0}^{N-1} f_j e^{2\pi i j n / N} (\Delta x) = (\Delta x) \sum_{j=0}^{N-1} f_j e^{2\pi i j n / N} \end{aligned}$$

inverse: $f_j = \frac{1}{N} \sum_{n=0}^{N-1} F_n \underbrace{e^{-2\pi i j n / N}}_{F_n}$

$$\left(\frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i j n / N} \left(\sum_{j'=0}^{N-1} f_{j'} e^{2\pi i j' n / N} \right) \right)$$

$$= \sum_{j=0}^{N-1} f_{j'} \left(\frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i (j-j') n / N} \right) = \sum_{j=0}^{N-1} f_{j'} \delta_{jj'} = f_{j'}$$

Note: We can write $F_n = \sum_{k=0}^{N-1} W^{nk} f_k$

$$W = e^{2\pi i/N}$$

$$F_n = \sum_{k=0}^{N-1} e^{2\pi i k n / N} f_k$$

$$= \sum_{k=0}^{N/2-1} e^{2\pi i n(2k)/N} f_{2k} + \sum_{k=0}^{N/2-1} e^{2\pi i n(2k+1)/N} f_{2k+1}$$

$$= \sum_{k=0}^{N/2-1} e^{2\pi i n k / (N/2)} f_{2k} + W^n \sum_{k=0}^{N/2-1} e^{2\pi i n k / (N/2)} f_{2k+1}$$

$$= F_n^e + W^n F_n^o$$

even
(length: $N/2$)

odd
(length: $N/2$)

Example:

$$F_0 \ F_1 \ F_2 \ F_3 \ F_4 \ F_5 \ F_6 \ F_7$$

$$F_0^e \ F_2^e \ F_4^e \ F_6^e$$

$$F_1^o \ F_3^o \ F_5^o \ F_7^o$$

$$F_0^{ee} \ F_4^{ee}$$

$$F_2^{eo} \ F_6^{eo}$$

$$F_1^{oe} \ F_5^{oe}$$

$$F_3^{oo} \ F_7^{oo}$$

Ordering: bit reversal

000	0	0	
001	1	4	(100)
010	2	2	
011	3	6	(110)
100	4	1	(001)
101	5	5	
110	6	3	(011)
111	7	7	

Typical structure of a "divide and conquer" algorithm.
 ↳ much faster than direct matrix multiplication

FFT can be generalized to higher dimensions.

FFT in practice:



$$x = \text{linspace}(x_{\min}, x_{\max} - \frac{x_D}{N}, N) \quad x_D = x_{\max} - x_{\min}$$

$$\text{dom} = \frac{2\pi}{x_D} \quad \omega = ([0:N/2, -N/2+1:-1]) * \text{dom}$$

↳ see example