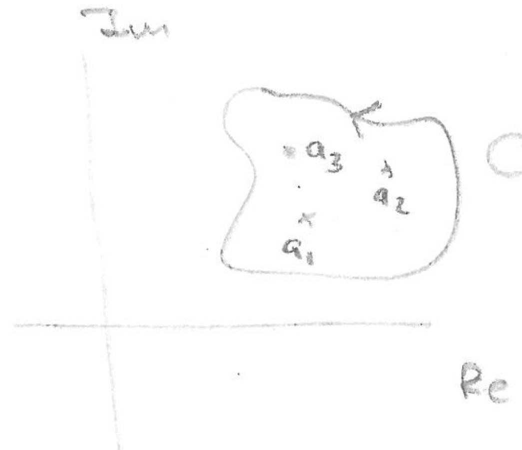
# Math Methods

## Lecture 7



**Residue Theorem:**




$$\oint_C f(z)\,dz = 2\pi i \sum_j \mathrm{Res}(f, a_j)$$

**Remarks:** If $a_j$ is a pole of m-th order, then

$$\mathrm{Res}(f, a_j) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{(m-1)}}{dz^{(m-1)}} \left[ (z-a_j)^m f(z) \right]$$

**Example:** $f(z) = \dfrac{e^z}{z(z+2)^2}$

$$\Rightarrow \mathrm{Res}(f, 0) = \frac{1}{4}, \quad \mathrm{Res}(f, -1) = \frac{1}{(2-1)!} \lim_{z \to -2} \frac{d}{dz}\left(\frac{e^z}{z}\right)$$

$$= \lim_{z \to -2}\left(\frac{e^z}{z} - \frac{e^z}{z^2}\right) = -\frac{3}{4} e^{-2}$$

(using the Laurent-Series is usually harder)

## Integrals:

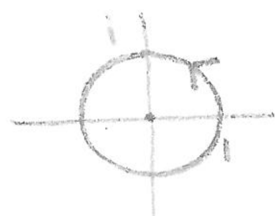
(1) $\displaystyle\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta.$

<u>Ex:</u> $\displaystyle J = \int_0^{2\pi} \frac{d\theta}{1-2p\cos\theta + p^2} \qquad (p<1)$

set $z = e^{i\theta} \Rightarrow z = \cos\theta + i\sin\theta$

$dz = i e^{i\theta} d\theta \qquad \bar{z}^1 = \cos\theta - i\sin\theta$ $\Big\} \Rightarrow \cos\theta = \frac{1}{2}(z + \bar{z}^1)$

$$J = \oint_C \frac{1}{1 - p(z+\bar{z}^1) + p^2} \frac{dz}{iz}$$




$C$ is the unit circle

$$= \frac{1}{i} \oint_C \frac{1}{z - pz^2 + p^2 z - p} dz = \frac{1}{i} \oint_C \frac{dz}{(1-pz)(z-p)}$$

pole: $z = p$ (inside the unit circle)

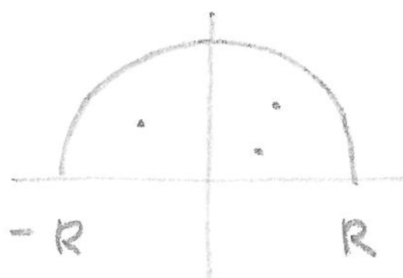
$\displaystyle \mathrm{Res}(f, p) = \frac{1}{1-p^2} \Rightarrow J = 2\pi i \cdot \frac{1}{i} \frac{1}{1-p^2}$

$$= \frac{2\pi}{1-p^2}$$

(2) Rational Function:

$$J = \int_{-\infty}^{\infty} f(x)\,dx \,, \quad \lim_{|x|\to\infty} x\, f(x) = 0$$

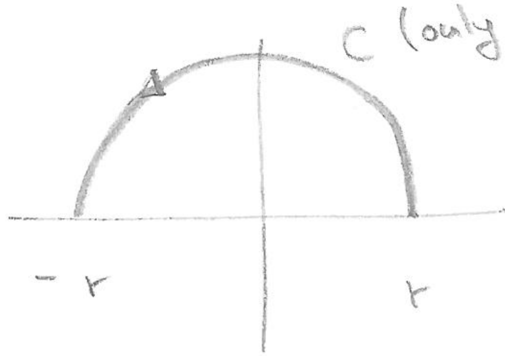
$$= 2\pi i \sum_j \mathrm{Res}(f, a_j)$$




where $a_j$ are the poles in the upper half-plane.

(3) If $\lim_{|z|\to\infty} f(z) = 0$ for $\mathrm{Im}\, z \geq 0$, then

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\, e^{ix}\, dx = 2\pi i \sum \mathrm{Res}\left(f(z)\, e^{iz}\right)$$




$$z = re^{i\theta}$$

$$M(r) = \max |f(re^{i\theta})|$$

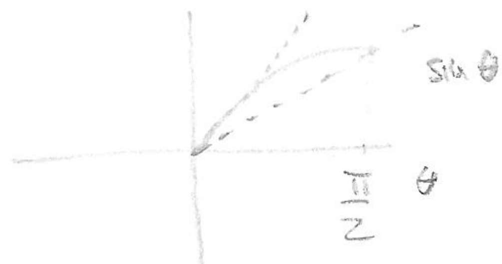
$$\left|\int_C f(z)\, e^{iz}\, dz\right| \leq M(r) \int_0^{\pi} e^{-r\sin\theta}\, r\, d\theta$$

Math Methods



$$= 2M(r)\int_0^{\pi/2} e^{-r\sin\theta}\, r\, d\theta \qquad \text{and} \quad \frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$$

$$\le 2M(r)\int_0^{\pi/2} e^{-r\theta\frac{2}{\pi}}\, r\, d\theta \qquad \text{for} \quad 0 \le \theta \le \frac{\pi}{2}$$




$$r\int_0^{\pi/2} e^{-r\theta\frac{2}{\pi}}\, d\theta = -\frac{\pi}{2}\, e^{-r\theta\frac{2}{\pi}}\Big|_0^{\pi/2}$$

$$= \frac{\pi}{2}\left(1 - e^{-r}\right)$$

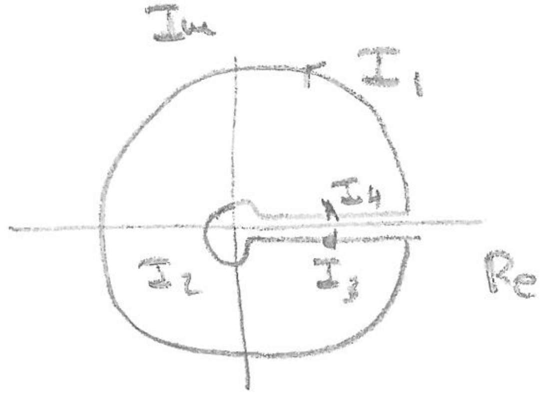
$$\Rightarrow \left|\int_C f(z)e^{iz}\,dz\right| \le \pi M(r)$$

(4) $\displaystyle\int_0^\infty \frac{f(x)}{x^q}\,dx$ , $q \in \mathbb{R}$, $0<q<1$

$f$ is a rational function, no pole on positive axis,

$\dfrac{f(z)}{z^{q-1}} \longrightarrow 0$ as $z \to 0$ and $z \to \infty$.

$$g(z) = \frac{f(z)}{z^\alpha}$$




branch cut along positive real axis

$(0 \le \arg z < 2\pi)$

$$\int_{C(r,\epsilon)} \frac{f(z)}{z^\alpha} dz = I_1 + I_2 + I_3 + I_4$$

$I_1, I_2 \to 0$ as $r \to \infty$, $\epsilon \to 0$

$$I_4 = \int_\epsilon^r \frac{f(x)}{x^\alpha} dx \qquad I_3 = -\int_\epsilon^r \frac{f(x)}{x^\alpha e^{2\pi i \alpha}} dx$$

$z^\alpha = |z|^\alpha e^{i \arg(z) \alpha}$ and for $I_3$ we have $\arg(z) = 2\pi$

$$\Rightarrow \int_{C(r,\epsilon)} \frac{f(z)}{z^\alpha} dz = \left(1 - e^{-2\pi i \alpha}\right) \int_\epsilon^r \frac{f(x)}{x^\alpha} dx$$

$$\Rightarrow \quad (1-e^{-2\pi i\alpha})\int_0^\infty \frac{f(x)}{x^\alpha}\,dx = 2\pi i \sum \text{Res}\left(\frac{f(z)}{z^\alpha}\right)$$

Ex:

$$\mathcal{I} = \int_0^\infty \frac{dx}{x^\alpha(1+x)} \qquad (0<\alpha<1)$$

$$f(z) = \frac{1}{1+z}, \quad \text{Res}\left(\frac{f(z)}{z^\alpha}, z=-1\right) = \frac{1}{e^{i\pi\alpha}}$$

$$\Rightarrow \quad \mathcal{I} = \frac{2\pi i}{e^{i\pi\alpha}}\,\frac{1}{1-e^{-2\pi i\alpha}} = \frac{\pi\, 2i}{e^{i\pi\alpha}-e^{-i\pi\alpha}} = \frac{\pi}{\sin\pi\alpha}.$$

# Math Methods

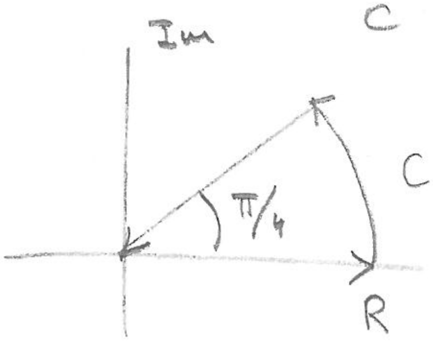
**Lecture 7**



## Fresnel Integrals:

$$\int_0^\infty \cos(kx^2)\,dx = \int_0^\infty \sin(kx^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2k}}$$

Consider $I = \oint_C e^{ikz^2}\,dz\,, \quad k>0$




$z = Re^{i\phi}$ $\left(0 \le \phi \le \frac{\pi}{4}\right)$ on arc

$z = x(1+i)$ on slanted segment

$$z^2 = x^2(1+i)^2 = x^2\cdot 2i$$

$$0 = \lim_{R\to\infty} \oint_C e^{ikz^2}\,dz = \int_0^\infty e^{ikx^2}\,dx + \underbrace{\lim_{R\to\infty}\left[\int_0^{\pi/4} e^{ikR^2 e^{2i\phi}}\, iRe^{i\phi}\,d\phi\right]}_{I_2} - (1+i)\int_0^\infty e^{-2kx^2}\,dx$$

$$\left| Re^{ikR^2 e^{2i\phi}} \right| = \left| R e^{ikR^2\cos(2\phi)}\, e^{-kR^2\sin(2\phi)} \right| = R e^{-kR^2 \sin(2\phi)}$$

but $\sin(2\phi) > 0$ in $0 < \phi \le \frac{\pi}{4}$ $\Rightarrow$ $I_2 \to 0$ as $R\to\infty$.

We know that $\int_{-\infty}^\infty e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$ $\Rightarrow$ $\int_0^\infty e^{-2kx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2k}}$

# Math Methods



$$\Rightarrow \int_0^\infty e^{ikx^2} = \frac{1+i}{2}\sqrt{\frac{\pi}{2k}} \qquad \square$$

## Argument principle:

let $f(z)$ be an analytic Function within a closed contour $C$ except at a finite number of poles If $f(z) \neq 0$ on $C$, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \underset{\substack{\uparrow \\ \#\text{ of} \\ \text{zeros}}}{N_0} - \underset{\substack{\uparrow \\ \#\text{ of} \\ \text{poles}}}{N_\infty}$$

(with multiplicity)

Note $g(z) = \dfrac{f'(z)}{f(z)} = \dfrac{d}{dz}\left(\ln(f(z))\right)$

Assume $f(z)$ has a zero of $n$-th order at $a$

$$\Rightarrow f(z) = (z-a)^n\left[c_1 + c_2(z-a) + \dots\right] \qquad c_1 \neq 0$$

$$\Rightarrow \ln f(z) = n\ln(z-a) + \ln \underbrace{\left[c_1 + c_2(z-a) + \dots\right]}_{f_1(z)}$$

$$\frac{f'(z)}{f(z)} = \frac{n}{z-a} + \frac{f_1'(z)}{f_1(z)}$$

$\Rightarrow$ $\mathrm{Res}(g, a) = n$

For poles, in a similar way $\dfrac{f'(z)}{f(z)} = -\dfrac{m}{z-b} + \dfrac{f_2'}{f_2}$

( pole of order $m$ $\Rightarrow$ $f(z) = \dfrac{1}{(z-b)^m} \underbrace{\left[ c_1 + c_2 (z-b) + \dots \right]}_{f_2}$

$\left( \ln f(z) = -m \ln (z-b) + \ln f_2 \right)$