Math Methods

Lecture 6

Complex Functions

Analytic Functions :

complex function: $f: D \to \mathbb{C}$, $D \subset \mathbb{C}$.

$z = x + iy$ $\quad f(x) = u(z) + i v(z)$

u and v are called real and imaginary part of f.

f is called <u>differentiable</u> at $z_0 \in D$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \quad z \in D \text{ exists}$$

(regardless of the manner in which z approaches z_0).

f is called <u>analytic</u> at $z_0 \in D$ if it is differentiable throughout a neighborhood of z_0.

$\boxed{\text{Theorem}}$ A function f is differentiable at $z_0 \in D$ if and only if

(i) the first-order partial derivatives of $u(x,y)$ and $v(x,y)$ exist and are continuous at z_0.

(ii) those derivatives at z_0 satisfy the <u>Cauchy</u> - <u>Riemann</u> relations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

<u>Proof of "$\Rightarrow$":</u>

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z} \right)$$

Take a path $\Delta z \to 0$ parallel to the real axis ($\Delta y = 0$)

$$\Rightarrow \quad f'(z_0) = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Take a path $\Delta z \to 0$ parallel to the imaginary axis ($\Delta x = 0$)

$$\Rightarrow \quad f'(z_0) = \lim_{i\Delta y \to 0} \left(\frac{\Delta u}{i\Delta y} + i \frac{\Delta v}{i\Delta y} \right) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \square$$

"$\Leftarrow$": see book.

Assume now that we can differentiate again (this is true, we'll prove it later)

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y}$$

(only if partial derivatives are continuous, but also true)

$\Rightarrow$ $\boxed{u_{xx} + u_{yy} = 0}$ $\Rightarrow$ u satisfies the Laplace equation

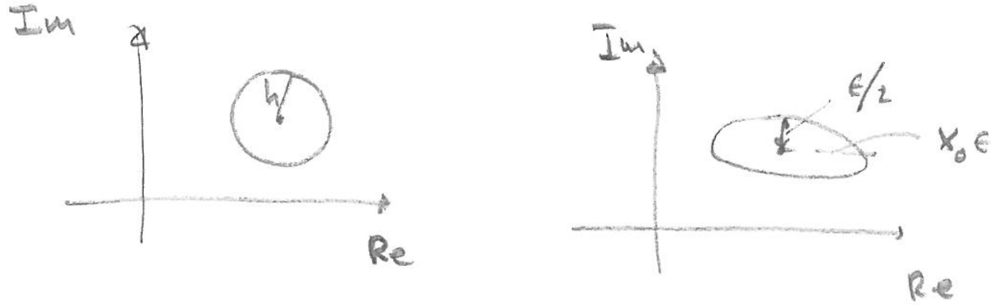
(those functions are called harmonic functions)
(Same is true for v)

Geometric Interpretation

$f(z) = x^2 + iy$ is not analytic except at $x = \frac{1}{2}$
($u_x = 2x \neq v_y = 1$). Set $h = \epsilon\cos\theta + i\epsilon\sin\theta$

$$f(z+h) = (x_0 + \epsilon\cos\theta)^2 + i\,(y_0 + \epsilon\sin\theta)$$

$$\approx x_0^2 + 2\epsilon x_0\cos\theta + iy_0 + i\epsilon\sin\theta + O(\epsilon^2)$$



circle ⟶ ellipse
(non-isotropic)

Analytic:

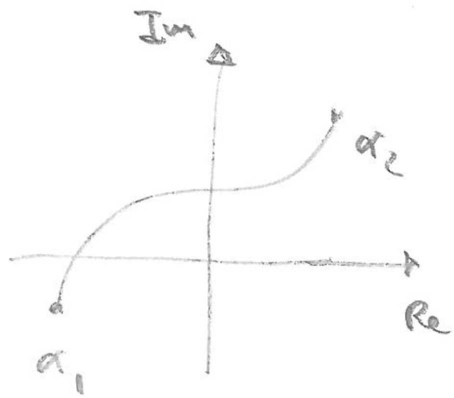
$$f(z) - f(z_0) \approx f'(z_0)\, h \Rightarrow |f(z) - f(z_0)| \approx |f'(z_0)|\,|h|$$

$$h = \epsilon e^{i\theta} \Rightarrow |h| = \epsilon$$

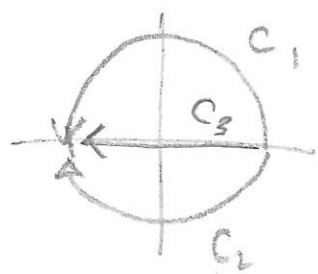
circle → isotropic
(circle + small corrections)

Complex integral

$$\int_{\alpha_1}^{\alpha_2} f(z)\,dz = \lim_{\substack{N\to\infty \\ \Delta z_i \to 0}} \sum_{i=1}^{N} f(z_i)\,\Delta z_i$$



Ex: $I = \oint_{C_i} z^* \, dz$



(i) $z = e^{i\theta}$ $\quad dz = i e^{i\theta} d\theta$

$$I(C_1) = \int_{C_1} z^* dz = \int_0^{\pi} e^{-i\theta}\, i e^{i\theta}\, d\theta = \pi i$$

$$I(C_2) = \int_{C_2} z^* dz = \int_0^{-\pi} e^{-i\theta}\, i\, e^{+i\theta}\, d\theta = -i\pi$$

$$I(C_3) = \int_{1}^{-1} x \, dx = 0$$

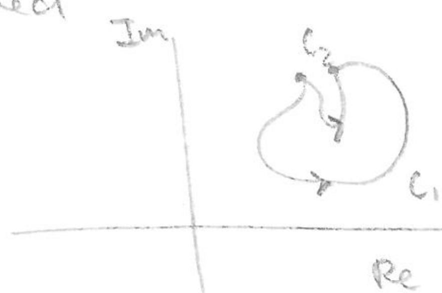
Usually: path-dependent. But if f is analytic within and on a closed contour C:

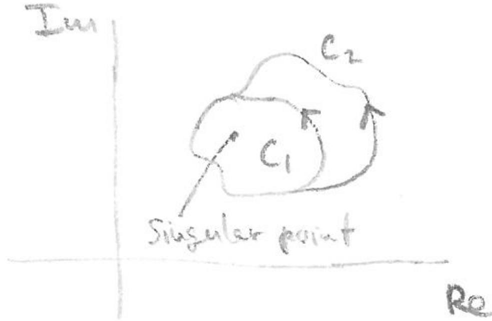
$$\oint_C f(z) \, dz = 0$$

$\Rightarrow$ If f is analytic in a region R and if contours C_1 and C_2 have the same endpoints

$$\int_{C_1} f \, dz = \int_{C_2} f \, dz$$

$\Rightarrow$ In an analytic region, the integration contour can be deformed





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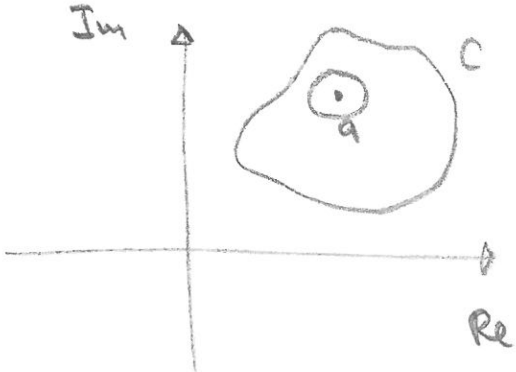
Cauchy Formula:
$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

if a is interior to C

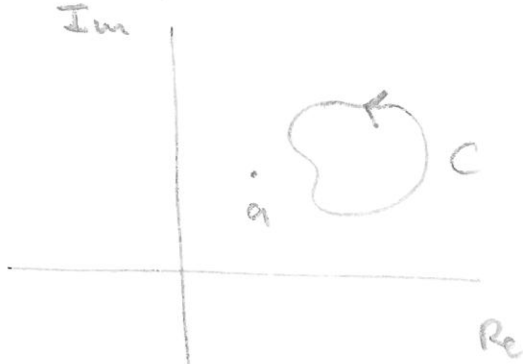


$$z = a + \epsilon e^{i\theta}$$

$$\oint_C \frac{f(z)}{z-a}\,dz$$

$$= \int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}\, i e^{i\theta}\, d\theta \to 2\pi i\, f(a) \text{ as } \epsilon \to 0.$$

Note: If C does not enclose a, then $\oint_C \frac{f(z)}{z-a}\,dz = 0$



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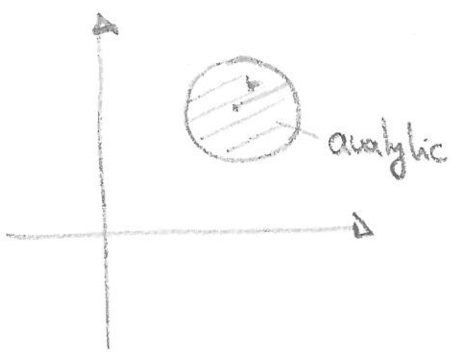
Cauchy: $$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\, dz$$

$\Rightarrow$ we can differentiate! $$f'(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)^2}\, dz$$

or $$f^{(n)}(a) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}}\, dz \quad \text{(Goursat formula)}$$

Cauchy inequality: $$\left| f^{(n)}(z) \right| \leq \frac{n!}{r^n} M(r)$$ within and on C



if f is analytic within and on a circle C with radius r, $M(r)$ is the maximum of $|f(z)|$ on C.

Proof:

$$\left| f^{(n)}(z) \right| \leq \frac{n!}{2\pi} \oint_C \frac{|f(\xi)|}{\left|(\xi - z)^{n+1}\right|} |d\xi| \leq \frac{n!}{2\pi r^{n+1}} r \cdot M(r)$$

$\square$

$$\left(\xi = z + r e^{i\theta} \right)$$

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<u>Note:</u> An analytic functions takes maxima and minima of $|f(z)|$ on C not within C.

<u>Liouville:</u> If f is an entire function (analytic on $\mathbb{C}$) and if $|f(z)|$ is bounded for all z, then f is constant.

<u>Proof:</u> $|f(z)| \leq M \Rightarrow |f^{(n)}(z)| \leq \frac{n!}{r^n} M$

$$\Rightarrow |f'(z)| \leq \frac{M}{r} \Rightarrow f'(z) = 0 \Rightarrow f = \text{const.}$$

<u>Note:</u> This yields a simple proof of the fundamental theorem of algebra:

Every non constant polynomial of degree n with complex coefficients has at least one zero in the complex plane.

Proof: Assume there was no zero $\Rightarrow \frac{1}{P(z)}$ is entire

If P is not constant $\Rightarrow \lim_{z \to \infty} \frac{1}{P} = 0 \Rightarrow$

$\frac{1}{P}$ is bounded $\Rightarrow$ constant $\lightning$.

$\square$

Series Representations:

Taylor: If f is analytic within and on the circle C of radius r around $z=a$, then

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k \qquad (|z-a| \leq r)$$

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - a)^{k+1}} d\xi$$

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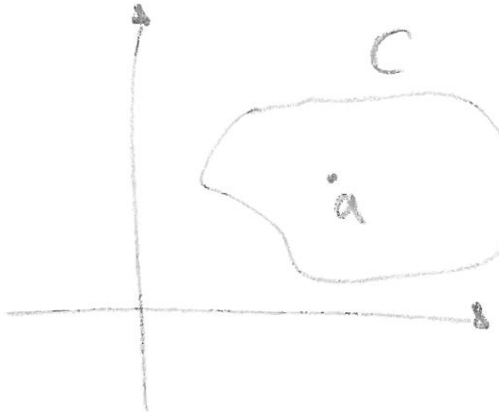
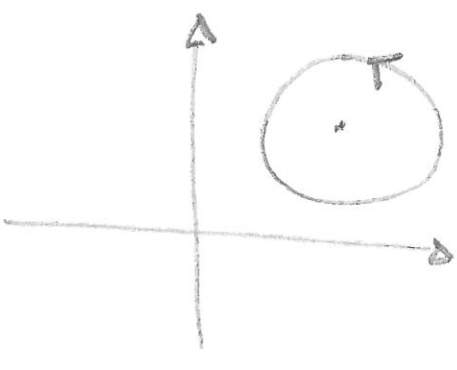
We can also expand around singular points:

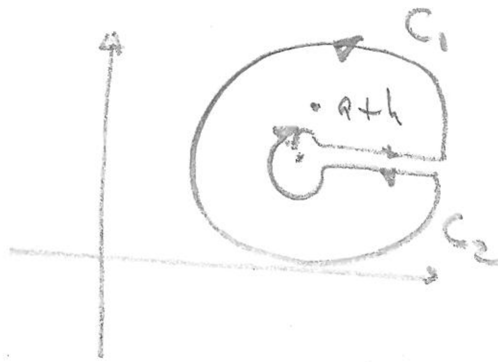


f analytic in and on C but not in a

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi \quad \text{(Laurent expansion)}$$





$$f(a+h) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a-h} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z-a-h}$$

$$- \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z-a-h}$$

$|z-a| > |h|$ on C_1 $\qquad$ $|z-a| < |h|$ on C_2

$$\frac{1}{z-a-h} = \frac{1}{z-a}\,\frac{1}{1-\frac{h}{z-a}} = \frac{1}{z-a}\sum_{n=0}^{\infty}\left(\frac{h}{z-a}\right)^n \quad \text{on } C_1$$

$$\frac{1}{z-a-h} = \frac{1}{h}\,\frac{1}{\frac{z-a}{h}-1} = \frac{-1}{h}\sum_{n=0}^{\infty}\left(\frac{z-a}{h}\right)^n \quad \text{on } C_2$$

$$\Rightarrow \quad f(a+h) = \frac{1}{2\pi i}\oint_{C_1}\sum_{n=0}^{\infty}\frac{h^n}{(z-a)^{n+1}}\,f(z)\,dz$$

$$+\ \frac{1}{2\pi i}\oint_{C_2}\sum_{n=1}^{\infty}\frac{(z-a)^{n-1}}{h^n}\,f(z)\,dz$$

$$f(a+h) = \sum_{n=-\infty}^{\infty} c_n h^n$$

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Note: Because of analyticity we can use e.g. $C = C_1$ for all values of n.

Example: (a) $f(z) = \dfrac{1}{z^2 - (2+i)z + 2i}$

Singular at $z = i$ and $z = 2$

$\Rightarrow$ Taylor series about $z=0$ $(|z|<1)$

two Laurant series about $z=0$, one for $1<|z|<2$, one for $|z|>2$

$$f(z) = \frac{1}{2-i}\left(\frac{1}{z-2} - \frac{1}{z-i}\right) \rightarrow \text{see book (HW)}$$

(b)

$$\frac{1}{(z-1)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right)$$

$$\frac{1}{1-z} = \begin{cases} \sum_{n=0}^{\infty} z^n, & |z|<1 \\ -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} & |z|>1 \end{cases}$$

$$\Rightarrow \frac{1}{(z-1)^2} = \begin{cases} \sum_{n=0}^{\infty} (n+1) z^n & |z|<1 \\ -\sum_{n=0}^{\infty} (n+1) z^{-(n+2)}, & |z|>1 \end{cases}$$

Singularities:

1. removable $f(z)$ is finite in a neighborhood of a.
2. pole of order m : $(z-a)^m f(z)$ is analytic at $z=a$ (but not $(z-a)^{m-1} f(z)$).
3. essential : Laurent series has an infinite number of terms involving negative powers of $(z-a)$.

Examples:

1. $f(z) = \dfrac{\sin z}{z}$

2. $\dfrac{1}{\sin z} = \dfrac{1}{z} + \dfrac{z}{6} + \dfrac{7}{360} z^3 + \cdots$

   simple pole at origin

3. $e^{1/z} = 1 + \dfrac{1}{z} + \dfrac{1}{2} \dfrac{1}{z^2} + \cdots$

   essential singularity at 0.

Residue Theorem.

Suppose that f has a pole of order m at a

$$\Rightarrow f(z) = \frac{c_{-m}}{(z-a)^m} + \frac{c_{-m+1}}{(z-a)^{m-1}} + \dots + \frac{c_{-1}}{(z-a)}$$

$$+ \sum_{n=0}^{\infty} c_n (z-a)^n \qquad (*)$$

we know that $c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$

$c_{-1} = \frac{1}{2\pi i} \oint f(z) dz$ is called residue of f at $z=a$. On the other had:

$g(z) = (z-a)^m f(z)$ is analytic in $z=a$

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!} (z-a)^n \overset{(*)}{=} \sum_{n=0}^{\infty} c_{n-m} (z-a)^n$$

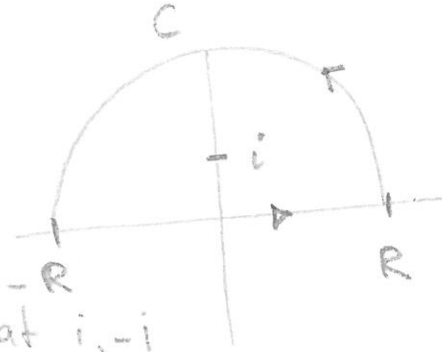
$\Rightarrow$ compare coefficients:

$n - m \stackrel{!}{=} -1$ or $n = m-1$

$$C_{-1} = \frac{g^{(m-1)}(a)}{(m-1)!} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{(m-1)}}{dz^{(m-1)}} \left((z-a)^m f(z) \right)$$

<u>very useful to compute integrals:</u>

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \pi$$



$f(z) = \dfrac{1}{1+z^2}$ has poles at $i, -i$

$$\oint_C f(z)\, dz = 2\pi i \cdot \text{res}(z=i) \quad \text{and} \quad \text{res}(z=i) = \lim_{z \to i} \frac{1}{z+i}$$

$$= \pi \qquad\qquad = \frac{1}{2i}$$

Only we need to show that the integral along the semi-circle vanishes for $R \to \infty$.