Math Methods

Lecture 5

Orthonormal Polynomials

Polynomial approximation: based on Weierstrass approximation theorem:

Theorem

If a function f is continuous on $[a,b]$, there exists a sequence of polynomials

$$G_n(x) = \sum_{k=0}^{n} c_k^{(n)} x^k$$

such that $G_n \to f$ uniformly on $[a,b]$

Gram-Schmidt orthogonalization:

$\{\varphi_i\}$ are linearly independent, normalizable

$\to$ we can construct an orthonormal set $\{Q_i\}$

$$Q_i = \frac{u_i}{\|u_i\|}$$

$$u_1 = \varphi_1 \quad , \quad u_i = \varphi_i - \sum_{k=1}^{i-1} \frac{(u_k, \varphi_i)}{(u_k, u_k)} u_k$$

The idea: The new u_i will be orthogonal to all u_k with $k = 1 \dots i-1$.

$$\left((u_k, u_i) = (u_k, \varphi_i) - \sum_{\tilde{k}=1}^{i-1} \frac{(u_{\tilde{k}}, \varphi_i)}{(u_{\tilde{k}}, u_{\tilde{k}})} (u_k, u_{\tilde{k}}) \right.$$

$$= (u_k, \varphi_i) - (u_k, \varphi_i)$$

$$\left. = 0 \quad \text{as desired} \right)$$

$\boxed{\underline{\text{Note:}}\ \frac{(u_k, u_{\tilde{k}})}{(u_{\tilde{k}}, u_{\tilde{k}})} = \delta_{k\tilde{k}} \text{ for } 1 \le k, \tilde{k} \le i-1}$

Example: Legendre Polynomials: use $[a,b] = [-1,1]$ and $\{1, x, x^2, \dots\}$

$$u_0 = P_0(x) = 1, \quad u_1 = x - \frac{(x, u_0)}{(u_0, u_0)} u_0 = x$$

$$(x, u_0) = \int_{-1}^{1} x \cdot 1 \, dx = 0$$

$$u_2 = x^2 - \frac{(x^2, u_0)}{(u_0, u_0)} u_0 - \frac{(x^2, u_1)}{(u_1, u_1)} u_1$$

$$= x^2 - \frac{1}{3} = \frac{1}{3}(3x^2 - 1)$$

Often $P_n(1) = 1 \Rightarrow P_0(x) = 1, \ P_1(x) = x, \ P_2(x) = \frac{1}{2}(3x^2 - 1)$

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General Formula:
$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}$$

$$= \cdots = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Rodrigues formula

Orthogonality follows from there: $n \geq m$

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \alpha \int_{-1}^{1} \underbrace{\left[\frac{d^n}{dx^n} (x^2-1)^n\right]}_{u'} \underbrace{\left[\frac{d^m}{dx^m} (x^2-1)^m\right]}_{v} dx$$

$$= \alpha \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \frac{d^m}{dx^m} (x^2-1)^m \Big|_{-1}^{1}$$

$$- \alpha \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \frac{d^{m+1}}{dx^{m+1}} (x^2-1)^m dx$$

$$\frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n = (x^2-1) \cdot \text{polynomial}$$

$\Rightarrow$ after n integrations by part.

$$\sim \int_{-1}^{1} (x^2-1) \underbrace{\frac{d^{m+n}}{dx^{m+n}} (x^2-1)^m}_{=0 \text{ for } n>m} dx$$

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$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n \underbrace{\frac{d^{2n}}{dx^{2n}}(x^2-1)^n}_{=(2n)!\ \left(\deg (x^2-1)^n = 2n\right)}\,dx$$

$$= \frac{(2n)!\,(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\,dx = \frac{2}{2n+1}$$

Remark: $$\int_{-1}^{1}(1-x^2)^n\,dx = 2^{2n+1}\int_0^1 t^n(1-t)^n\,dt$$

$$= 2^{2n+1}B(n+1,n+1) = 2^{2n+1}\frac{\Gamma(n+1)^2}{\Gamma(2n+2)} = 2^{2n+1}\frac{(n!)^2}{(2n+1)!}$$

As $$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\Gamma(x) = \int_0^\infty e^{-t}\,t^{x-1}\,dt$$

Fourier Series

Weierstrass Theorem for two variables

$$g_N(x,y) = \sum_{n,m=0}^{N} a_{nm}^{(N)} x^n y^n$$

Polar coordinates $e^{i\theta} = \cos\theta + i\sin\theta$

$$\cos^n\theta = \left[\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right]^n, \quad \sin^n\theta = \left[\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right]^n$$

on unit circle:

$$f_M(x) = \sum_{n=-M}^{M} \frac{c_n^{(M)}}{(2\pi)^{1/2}} e^{inx} \qquad M = 2N$$

One can show that the functions $\{F_n\}$ with

$$F_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$$ are a complete ONB of $L^2_{per}(-\pi,\pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n F_n(x), \quad c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

(Fourier Series)

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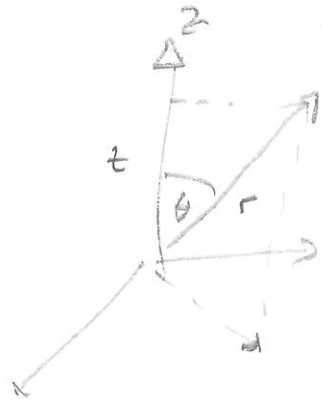
Spherical Harmonic Functions:

$$g_M(r) = \sum_{j,k,n=0}^{M} a_{jkn}^{(M)}\, x^j y^k z^n$$

$$u = x + iy = r \sin\theta\, e^{i\varphi}$$

$$v = x - iy = r \sin\theta\, e^{-i\varphi}$$

$$w = r \cos\theta$$



$$\Rightarrow\ g_M(r) = \sum_{\alpha,\beta,\gamma=0}^{M} b_{\alpha\beta\gamma}^{(M)}\, u^\alpha v^\beta w^\gamma$$

$$= \sum_{l=0}^{3M} r^l \sum_{(\alpha,\beta,\gamma)} b_{\alpha\beta\gamma}^{(M)}\, e^{i(\alpha-\beta)\varphi} \sin^{\alpha+\beta}\theta \cos^\gamma\theta$$

$$\hookrightarrow \alpha+\beta+\gamma = l$$

unit sphere: $|r| = 1$, $m = \alpha - \beta$

$$g_M(\theta,\varphi) = \sum_{l=0}^{3M} \sum_{(\alpha,\beta,\gamma)} b_{\alpha\beta\gamma}^{(M)}\, e^{im\varphi} \sin^{\alpha+\beta-|m|}\theta \cos^\gamma\theta \sin^{|m|}\theta$$

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$$\sin^{\alpha+\beta-|m|}\theta \cos^{\gamma}\theta = (1-\cos^2\theta)^{(\alpha+\beta-|m|)/2}\cos^{\gamma}\theta$$
$$\equiv f_{\ell m}(\cos\theta)$$

$\Rightarrow$ polynomial in $\cos\theta$ of degree $\alpha+\beta+\gamma-|m| = \ell-|m|$

and $\alpha+\beta-|m|$ is even

$$\Rightarrow \quad g_M(\theta,\varphi) = \sum_{\ell=0}^{3M}\sum_m b_{\ell m}^{(M)} e^{im\varphi}\sin^{|m|}\theta\, f_{\ell m}(\cos\theta)$$

($\alpha+\beta-m = m-|m|+2\beta$ as $m=\alpha-\beta$

($m-|m|$ is always even)

one can show that $\alpha+\beta+\gamma=\ell\geq 0$ yields

$-\ell\leq m\leq\ell$

Set $Y_{\ell m}(\theta,\varphi) = e^{im\varphi}\sin^{|m|}\theta\, f_{\ell m}(\cos\theta)$

$\uparrow$ spherical harmonics

$\uparrow$ polynomial in $\cos\theta$ of degree $\ell-|m|$

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$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \, Y^*_{\ell' m'}(\theta,\varphi) \, Y_{\ell m}(\theta,\varphi)$$

$$= \delta_{\ell\ell'} \delta_{mm'}$$

$$Y_{\ell m}(\theta,\varphi) = (-1)^m \left(\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right)^{1/2} P_\ell^m(\cos\theta) \, e^{im\varphi}, \quad m \geq 0$$

$$Y_{\ell,-m}(\theta,\varphi) = (-1)^m \, Y^*_{\ell m}(\theta,\varphi), \quad m \geq 0$$

$$P_\ell^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$$

associated Legendre Functions.

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General Rodrigues Formula;

$$Q_n(x) = \frac{1}{k_n\, w(x)} \frac{d^n}{dx^n} \left(w(x)\, s^n(x) \right)$$

1. Q_1 is a first degree polynomial
2. s is a polynomial of degree no more than 2 with real roots
3. w is real, positive $\quad w(a)s(a) = w(b)s(b) = 0$

$\Rightarrow$ Q_n is a polynomial in x of degree n,

$$\int_a^b p_m(x)\, Q_n(x)\, w(x)\, dx = 0 \quad (m < n)$$

arbitrary polynomial of degree $m < n$.

Classification:

$$Q_1(x) = -\frac{x}{k_1}$$

$$\Rightarrow \quad -\frac{x}{k_1} = \frac{1}{k_1 w(x)} \frac{d}{dx}(ws)$$

$$-x = \frac{1}{w}(w's + s'w) = \frac{w'}{w}s + s'$$

$$-x - s' = \frac{w'}{w}s \quad \text{or} \quad \frac{1}{w}w' = -\frac{x+s'}{s}$$

1. $S(x) = a$

$$\Rightarrow \quad \frac{1}{w}w' = -\frac{x}{a} \quad \Rightarrow \quad w(x) = A\, e^{-x^2/2a}$$

For b/c set $a = -\infty$, $b = +\infty$, choose $a = \frac{1}{2}$, $A = 1$

$\Rightarrow$ Hermite polynomials (or $a = 1$)

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \delta_{nm}$$

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Recurrence formula:

$$Q_{n+1}(x) = (a_n x + b_n)\, Q_n(x) - c_n\, Q_{n-1}$$

a_n, b_n, c_n depend on the class of polynomials considered.

Proof: Book (use orthogonality relation)

Example: Hermite: $H_{n+1}(x) = 2x H_n(x) - 2n\, H_{n-1}(x)$

Roots of orthogonal polynomials

can be related to eigenvectors of a particular matrix

ODE:
$$\frac{d}{dx}\left(s w \frac{dQ_n}{dx} \right) = -\lambda_n w Q_n$$

$$\lambda_n = -n \left(k_1 \frac{dQ_1}{dx} + \frac{n-1}{2} \frac{d^2 s}{dx^2} \right)$$

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Generating function:

$$f(t) = \sum_k f_k t^k$$

$$\Rightarrow \quad f_n = \frac{1}{n!} \frac{d^n f(t)}{dt^n}$$

Orthogonal polynomials: $g(t,x) = \sum_{n=0}^{\infty} A_n Q_n(x) t^n$

Cauchy: $$\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi - z)^{n+1}}$$

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

$$= (-1)^n e^{x^2/2} \frac{n!}{2\pi i} \oint_C \frac{e^{-\xi^2/2} d\xi}{(\xi - x)^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \frac{e^{x^2/2}}{2\pi i} \oint_C e^{-\xi^2/2} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(\xi - x)^{n+1}}$$

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$$\sum_{n=0}^{\infty} \frac{1}{\xi - x}\left(\frac{-t}{\xi - x}\right)^n = \frac{1}{\xi - x}\,\frac{1}{1+\frac{t}{\xi - x}} = \frac{1}{\xi - x + t}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \frac{e^{x^2/2}}{2\pi i} \oint_C \frac{e^{-\xi^2/2}}{\xi - x + t}$$

$$= \frac{e^{x^2/2}}{2\pi i} \cdot 2\pi i \, e^{-(x-t)^2/2} = e^{-t^2/2} e^{tx}$$

Chebyshev Polynomials

$$T_n(x) = \cos\left(n \cos^{-1}(x)\right)$$

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$$

$$\int_{-1}^{1} T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \neq 0 \end{cases}$$

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Generating functions (II)

Legendre polynomials: Following recursion

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad (P_{-1} = 0)$$

$$g(t,x) = \sum_{n=0}^{\infty} P_n(x) t^n \rightarrow$$ can we find an ODE for g?

$$\frac{\partial g}{\partial t} = \sum_{n=1}^{\infty} n P_n(x) t^{n-1} = \sum_{n=0}^{\infty} (n+1) P_{n+1}(x) t^n$$

$$= \sum_{n=0}^{\infty} \left[(2n+1) x P_n(x) - n P_{n-1}(x)\right] t^n$$

$$\frac{\partial g}{\partial t} = \overbrace{xg + 2tx\frac{\partial g}{\partial t}} - \sum_{n=0}^{\infty} \underbrace{n}_{(n = n-1+1)} P_{n-1}(x) t^n$$

$$= xg + 2tx\frac{\partial g}{\partial t} - tg - t^2 \frac{\partial g}{\partial t}$$

$$\frac{1}{g}\frac{\partial g}{\partial t} = \frac{x-t}{1-2tx+t^2} \quad \text{and} \quad g(0,x) = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)\, t^n = \frac{1}{\sqrt{1-2tx+t^2}} = g(x,t)$$

$$\left[\quad g(x,t) = \frac{1}{\sqrt{1-2tx+t^2}} \quad \Rightarrow \quad \frac{\partial g}{\partial t} = -\frac{1}{2}\frac{1}{\left(\sqrt{1-2tx+t^2}\right)^3}(-2x+2t)\right.$$

$$\left. = \frac{x-t}{(1-2tx+t^2)} \cdot g \quad \right]$$

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More on Chebyshev: (see Trefethen, Spectral Methods in Matlab)

Chebyshev points: $x_j = \cos\left(\frac{j\pi}{N}\right) \quad j = 0, 1, \dots, N$

p: polynomial $p(x_j) = v_j$ for $0 \le j \le N$

Derivative: $w = D_N v \qquad D_N : (N+1)^2$ - matrix

Interpolation: (Lagrange) If (x_j, v_j) are given:

$$L(x) = \sum_{j=0} v_j \, \ell_j(x) \qquad \ell_j(x) = \prod_{\substack{0 \le m \le N \\ m \neq j}} \frac{x - x_m}{x_j - x_m}$$

$N = 1$: $\quad x_0 = 1, \; x_1 = -1$

$$p(x) = \frac{1}{2}(1+x)v_0 + \frac{1}{2}(1-x)v_1$$

$$p'(x) = \frac{1}{2}v_0 - \frac{1}{2}v_1$$

$$\Rightarrow \quad D_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$N=2$: $p(x) = \frac{1}{2} x(1+x) v_0 + (1-x)(1+x) v_1 + \frac{1}{2} x(x-1) v_2$

$(1, 0, -1)$

$p'(x) = \left(x + \frac{1}{2}\right) v_0 - 2x v_1 + (x - \frac{1}{2} v_2)$

$p'(x_0) = p'(1) = \frac{3}{2} v_0 - 2 v_1 + \frac{1}{2} v_2$

$p'(x_1) = p'(0) = \frac{1}{2} v_0 - \frac{1}{2} v_2$

$p'(x_2) = p'(-1) = -\frac{1}{2} v_0 + 2 v_1 - \frac{3}{2} v_2$

$$\Rightarrow \quad D_2 = \begin{pmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{3}{2} \end{pmatrix}$$