

Orthonormal Polynomials

Polynomial approximation: based on Weierstrass approximation theorem:

Theorem

If a function f is continuous on $[a, b]$, there exists a sequence of polynomials

$$G_n(x) = \sum_{k=0}^n c_k^{(n)} x^k$$

such that $G_n \rightarrow f$ uniformly on $[a, b]$

Gram-Schmidt orthogonalization:

$\{\varphi_i\}$ are linearly independent, normalizable

$\rightarrow$ we can construct an orthonormal set $\{Q_i\}$

$$Q_i = \frac{u_i}{\|u_i\|}$$

$$u_1 = \varphi_1, \quad u_i = \varphi_i - \sum_{k=1}^{i-1} \frac{(u_k, \varphi_i)}{(u_k, u_k)} u_k$$

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The idea: The new u_i will be orthogonal to all u_k with $k=1 \dots i-1$.

$$\begin{aligned}
 (u_k, u_i) &= (u_k, \varphi_i) - \sum_{\tilde{k}=1}^{i-1} \frac{(u_{\tilde{k}}, \varphi_i)}{(u_{\tilde{k}}, u_{\tilde{k}})} \underbrace{(u_k, u_{\tilde{k}})}_{\delta_{k\tilde{k}}} \\
 &= (u_k, \varphi_i) - (u_k, \varphi_i) \\
 &= 0 \quad \text{as desired}
 \end{aligned}$$

Example: Legendre Polynomials: use $[a, b] = [-1, 1]$ and $\{1, x, x^2, \dots\}$

$$\begin{aligned}
 u_0 = P_0(x) &= 1, \quad u_1 = x - \frac{(x, u_0)}{(u_0, u_0)} u_0 \\
 &= x, \quad (x, u_0) = \int_{-1}^1 x \cdot 1 \, dx = 0
 \end{aligned}$$

$$u_2 = x^2 - \frac{(x^2, u_0)}{(u_0, u_0)} u_0 - \frac{(x^2, u_1)}{(u_1, u_1)} u_1$$

$$= x^2 - \frac{1}{3} = \frac{1}{3}(3x^2 - 1)$$

Often $P_n(1) = 1 \Rightarrow P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1)$

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General Formula: $P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}$

$$= \dots = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

↑
Rodrigues formula

Orthogonality follows from there: $n \neq m$

$$\int_{-1}^1 P_n(x) P_m(x) dx = \alpha \int_{-1}^1 \left[\frac{d^n}{dx^n} (x^2-1)^n \right] \left[\frac{d^m}{dx^m} (x^2-1)^m \right] dx$$

$$= \alpha \left. \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \frac{d^m}{dx^m} (x^2-1)^m \right|_{-1}^1$$

$$= \alpha \int_{-1}^1 \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \frac{d^{m+1}}{dx^{m+1}} (x^2-1)^m dx$$

$$\frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n = (x^2-1) \cdot \text{polynomial}$$

⇒ after n integrations by part.

$$\sim \int_{-1}^1 (x^2-1) \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^m dx$$

= 0 for $n > m$

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$$\int_{-1}^1 T_n^2(x) dx = \frac{(-1)^n}{2^{2n} (n!)^2} \int_{-1}^1 (x^2-1)^n \frac{d^{2n}}{dx^{2n}} (x^2-1)^n dx$$

$$= (2n)! \quad (\deg (x^2-1)^n = 2n)$$

$$= \frac{(2n)! (-1)^n}{2^{2n} (n!)^2} \int_{-1}^1 (x^2-1)^n dx = \frac{2}{2n+1}$$

Remark: $\int_{-1}^1 (1-x^2)^n dx = 2^{2n+1} \int_0^1 t^n (1-t)^n dt$

$$= 2^{2n+1} B(n+1, n+1) = 2^{2n+1} \frac{\Gamma(n+1)^2}{\Gamma(2n+2)} = 2^{2n+1} \frac{(n!)^2}{(2n+1)!}$$

As $B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

Fourier Series

Weierstrass Theorem for two variables

$$g_N(x, y) = \sum_{n, m=0}^N a_{nm}^{(N)} x^n y^m$$

Polar coordinates $e^{i\theta} = \cos \theta + i \sin \theta$

$$\cos^4 \theta = \left[\frac{1}{2} (e^{i\theta} + e^{-i\theta}) \right]^4, \quad \sin^4 \theta = \left[\frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \right]^4$$

on unit circle:

$$f_M(x) = \sum_{n=-M}^M \frac{c_n^{(M)}}{(2\pi)^{1/2}} e^{inx} \quad M=2N$$

One can show that the functions $\{F_n\}$ with

$$F_n(x) = \frac{e^{inx}}{\sqrt{2\pi}} \text{ are a complete ONB of } L^2_{\text{per}}(-\pi, \pi)$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n F_n(x), \quad c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \bar{e}^{-inx} dx$$

(Fourier Series)

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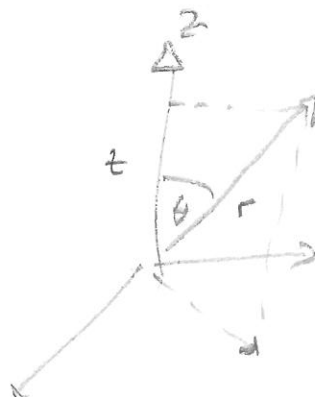
Spherical Harmonic Functions;

$$g_M(r) = \sum_{j,k,l=0}^M a_{jkl}^{(M)} x^j y^k z^l$$

$$u = x + iy = r \sin \theta e^{i\varphi}$$

$$v = x - iy = r \sin \theta e^{-i\varphi}$$

$$w = r \cos \theta$$



$$\Rightarrow g_M(r) = \sum_{\alpha,\beta,\gamma=0}^M b_{\alpha\beta\gamma}^{(M)} u^\alpha v^\beta w^\gamma$$

$$= \sum_{l=0}^{3M} r^l \sum_{(\alpha,\beta,\gamma)} b_{\alpha\beta\gamma}^{(M)} e^{i(\alpha-\beta)\varphi} \sin^{\alpha+\beta} \theta \cos^\gamma \theta$$

$$\alpha + \beta + \gamma = l$$

unit sphere: $|r|=1$, $m = \alpha - \beta$

$$g_M(\theta, \varphi) = \sum_{l=0}^{3M} \sum_{(\alpha,\beta,\gamma)} b_{\alpha\beta\gamma}^{(M)} e^{im\varphi} \sin^{\alpha+\beta-|m|} \theta \cos^\gamma \theta \sin^{|m|} \theta$$

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$$\sin^{\alpha+\beta-|m|} \theta \cos^{\gamma} \theta = (1 - \cos^2 \theta)^{(\alpha+\beta-|m|)/2} \cos^{\gamma} \theta$$

$\Rightarrow$ polynomial in $\cos \theta$ of degree $\alpha+\beta+\gamma-|m| = l-|m|$

and $\alpha+\beta-|m|$ is even

$$\Rightarrow g_M(\theta, \varphi) = \sum_{l=0}^{3M} \sum_m b_{lm}^{(M)} e^{im\varphi} \sin^{|m|} \theta P_{lm}(\cos \theta)$$

$$(\alpha+\beta-m = m-|m| + 2\beta \text{ as } m = \alpha-\beta$$

($m-|m|$ is always even)

one can show that $\alpha+\beta+\gamma = l \geq 0$ yields

$$-l \leq m \leq l$$

$$\text{Set } Y_{lm}(\theta, \varphi) = e^{im\varphi} \sin^{|m|} \theta P_{lm}(\cos \theta)$$

$\uparrow$
spherical harmonics

$\uparrow$
polynomial in $\cos \theta$
of degree $l-|m|$

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$$\int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) \\ = \delta_{ll'} \delta_{mm'}$$

$$Y_{lm}(\theta, \varphi) = (-1)^m \left(\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right)^{1/2} P_l^m(\cos\theta) e^{im\varphi}, \quad m \geq 0$$

$$Y_{l,-m}(\theta, \varphi) = (-1)^m Y_{lm}^*(\theta, \varphi), \quad m \geq 0$$

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

↑ associated Legendre Functions.

⑨

Grewal Rodrigues Formula,

$$Q_n(x) = \frac{1}{k_n w(x)} \frac{d^n}{dx^n} (w(x) S^n(x))$$

1. Q_1 is a first degree polynomial
2. S is a polynomial of degree no more than 2 with real roots
3. w is real, positive $w(a)S(a) = w(b)S(b) = 0$

$\Rightarrow Q_n$ is a polynomial in x of degree n ,

$$\int_a^b p_m(x) Q_n(x) w(x) dx = 0 \quad (m < n)$$

$\uparrow$
arbitrary polynomial of degree $m < n$,

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Classification:

$$Q_1(x) = -\frac{x}{k_1}$$

$$\Rightarrow -\frac{x}{k_1} = \frac{1}{k_1 w(x)} \frac{d}{dx} (ws)$$

$$-x = \frac{1}{w} (w's + s'w) = \frac{w'}{w} s + s'$$

$$-x - s' = \frac{w'}{w} s \quad \text{or} \quad \frac{1}{w} w' = -\frac{x + s'}{s}$$

1. $S(x) = \alpha$

$$\Rightarrow \frac{1}{w} w' = -\frac{x}{\alpha} \Rightarrow w(x) = A e^{-x^2/2\alpha}$$

For b/c set $a = -\infty$, $b = +\infty$, choose $\alpha = \frac{1}{2}$, $A = 1$

$\Rightarrow$ Hermite polynomials (or $\alpha = 1$)

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = S_{mn}$$

Recurrence formula:

$$Q_{n+1}(x) = (a_n x + b_n) Q_n(x) - c_n Q_{n-1}(x)$$

a_n, b_n, c_n depend on the class of polynomials considered.

Proof: Book (use orthogonality relation)

Example: Hermite: $H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$

Roots of orthogonal polynomials

can be related to eigenvectors of a particular matrix

ODE:
$$\frac{d}{dx} \left(s w \frac{dQ_n}{dx} \right) = -\lambda_n w Q_n$$

$$\lambda_n = -n \left(k_1 \frac{dQ_1}{dx} + \frac{n-1}{2} \frac{d^2 s}{dx^2} \right)$$

Generating Function:

$$f(t) = \sum_k f_k t^k$$

$$\Rightarrow f_n = \frac{1}{n!} \frac{d^n}{dt^n} f(t)$$

Orthogonal polynomials: $g(t, x) = \sum_{n=0}^{\infty} A_n Q_n(x) t^n$

Cauchy: $\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\xi) d\xi}{(\xi - z)^{n+1}}$

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

$$= (-1)^n e^{x^2/2} \frac{n!}{2\pi i} \oint_C \frac{e^{-\xi^2/2} d\xi}{(\xi - x)^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \frac{e^{x^2/2}}{2\pi i} \oint_C \frac{e^{-\xi^2/2}}{(\xi - x)^{n+1}} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(\xi - x)^{n+1}}$$

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$$\sum_{n=0}^{\infty} \frac{1}{z-x} \left(\frac{-t}{z-x} \right)^n = \frac{1}{z-x} \frac{1}{1 + \frac{t}{z-x}} = \frac{1}{z-x+t}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \frac{e^{x^2/2}}{2\pi i} \oint_C \frac{e^{-z^2/2}}{z-x+t} dz$$

$$= \frac{e^{x^2/2}}{2\pi i} \cdot 2\pi i e^{-(x-t)^2/2} = e^{-t^2/2} e^{tx}$$

Chebyshev Polynomials

$$T_n(x) = \cos(n \cos^{-1}(x))$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

$$\int_{-1}^1 T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & n \neq m \\ \pi & n=m=0 \\ \frac{\pi}{2} & n=m \neq 0 \end{cases}$$

Generating Functions (II)

Legendre polynomials: Following recursion

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad (P_{-1} = 0)$$

$$g(t, x) = \sum_{n=0}^{\infty} P_n(x) t^n \rightarrow \text{can we find an ODE for } g?$$

$$\begin{aligned} \frac{\partial g}{\partial t} &= \sum_{n=1}^{\infty} n P_n(x) t^{n-1} = \sum_{n=0}^{\infty} (n+1) P_{n+1}(x) t^n \\ &= \sum_{n=0}^{\infty} [(2n+1)x P_n(x) - n P_{n-1}(x)] t^n \end{aligned}$$

$$\frac{\partial g}{\partial t} = \overbrace{xg + 2tx \frac{\partial g}{\partial t}} - \sum_{n=0}^{\infty} \underbrace{n P_{n-1}(x) t^n}_{(n = \overbrace{n-1}^{n-1} + 1)}$$

$$= xg + 2tx \frac{\partial g}{\partial t} - tg - t^2 \frac{\partial g}{\partial t}$$

$$\frac{1}{g} \frac{\partial g}{\partial t} = \frac{x-t}{1-2tx+t^2} \quad \text{and} \quad g(0, x) = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1-2tx+t^2}}$$

$$\left[\begin{aligned} g(x,t) &= \frac{1}{\sqrt{1-2tx+t^2}} \Rightarrow \frac{\partial g}{\partial t} = -\frac{1}{2} \frac{1}{\left(\sqrt{1-2tx+t^2}\right)^3} (-2x+2t) \\ &= \frac{x-t}{\sqrt{1-2tx+t^2}} \cdot g \end{aligned} \right]$$