Math Methods

Lecture 4

Ordinary Differential Equations (ODEs)

Implicit: $F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0$

n: order of ODE

linear ODE: $a_n(x) y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_0(x) y(x) = q(x)$

In general: An ODE can have many solutions

$$y' = e^x \to y(x) = e^x + C$$

$$y''' = e^x \to y(x) = e^x + c_1 x^2 + c_2 x + c_3$$

- Not all ODEs have infinitely many solutions
  $(y'')^2 + y^2 = 0$ has only $y = 0$ as solution
- For an n-th-order ODE we do not always have n integration constants $(y'-y)(y'-2y) = 0$

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Existence Theorem for first-order ODEs

(*) $$y'(x) = f(x, y(x)), \quad y(x_0) = y_0$$

f : continuous, real valued

(*) $\Rightarrow$ $$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$ an integral equation.

Now we can use Picard iteration: $\varphi_0 = y_0$, $\varphi_1 = y_0 + \int_{x_0}^{x} f(t, \varphi_0(t))\,dt$

$$\varphi_n = y_0 + \int_{x_0}^{x} f(t, \varphi_{n-1}(t))\,dt$$

This works if f is Lipschitz :

$$| f(x, y(x)) - f(x, z(x)) | \leq K \, | y(x) - z(x) |$$

$\Rightarrow$ We have existence and uniqueness.

Note: $y' = \sqrt{|y(x)|}$ $\quad y(0) = 0$ has two solutions

$$y = 0 \quad \text{and} \quad y(x) = \begin{cases} x^2/4 & x \geq 0 \\ -x^2/4 & x < 0 \end{cases}$$

but is not Lipschitz around $y = 0$

$$\frac{f(x,y_2)-f(x,y_1)}{|y_2-y_1|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}} \quad \text{For } y_2 \geq 0, \; y_1 = 0$$

Sturm-Liouville Problems

A Sturm-Liouville equation is a second-order ODE of the form

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda w(x)y = 0 \qquad (*)$$

λ is a parameter, p, q, w are real-valued functions, $p(x) > 0$, $w(x) > 0$. w is called <u>weight</u> function

$$L = \frac{1}{w(x)}\left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right]$$

is the Sturm-Liouville operator $(*) \Leftrightarrow Ly = -\lambda y$

A Sturm-Liouville system is $(*)$ and separated boundary conditions: $y(a) = \alpha y'(a)$, $y(b) = \beta y'(b)$ with $\alpha, \beta \in \mathbb{R}$.

<u>Example:</u> $y'' + \lambda y = 0 \quad 0 \leq x \leq \pi$, $y(0) = 0$, $y(\pi) = 0$

$\Rightarrow y_n(x) = \sin nx$, $\lambda_n = n^2$

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Remark: Any second-order ODE can be transformed.

$$a(x)\,y'' + b(x)\,y' + c(x)\,y + \lambda e(x)\,y = 0$$

can be transformed by multiplying the factor

$$\xi(x) = \exp\left[\int^x \frac{b(s) - a'(s)}{a(s)}\,ds\right]$$

which yields a Sturm-Liouville form

$$(\xi a y')' + \xi c y + \lambda \xi e y = 0$$

Ex: Hermite equation

$$y'' - 2xy' + 2ay = 0$$

$$a(x) = 1, \quad b(x) = -2x \implies \xi(x) = e^{-x^2}$$

$$(y'' - 2xy')\,e^{-x^2} = -2a\,e^{-x^2}y$$

$$\left(y' e^{-x^2}\right)' = -2a\,e^{-x^2}y$$

is of Sturm-Liouville form

Ex: Bessel equation: $x^2y'' + xy' + (x^2 - n^2)\,y = 0$

Use transform $x \to kx$.

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Sturm-Liouville : self-adjoint

A Sturm-Liouville operator is self-adjoint on $[a,b]$ if any two eigenfunctions satisfy the boundary condition $[p y_i^* y_j']_a^b = 0$. Note: Let $\omega = 1$.

Proof: $(y_i, Ly_j) = -\int_a^b y_i^* (p y_j')' dx - \int_a^b y_i^* q y_j \, dx$

$= -[y_i^* (p y_j')]_a^b + \int_a^b y_i^{*\prime} (p y_j') dx - \int_a^b y_i^* q y_j \, dx$

$= +[(y_i^{*\prime} p) y_j]_a^b - \int_a^b (y_i^{*\prime} p)' y_j \, dx - \int_a^b y_i^* q y_j \, dx$

$= (Ly_i, y_j)$ $\square$

Note: All eigenvalues are real.
Eigenfunctions to distinct eigenvalues are orthogonal.

For general ω: use weighted inner product.

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Systems of ODEs:

General: implicit, usually

$$\left.\begin{aligned} y_1'(x) &= f_1(x, y, \dots, y_n) \\ &\vdots \\ y_n'(x) &= f_n(x, y, \dots, y_n) \end{aligned}\right\} \quad \mathbf{y}' = \mathbf{f}(x, \mathbf{y})$$

Reducing the order: $\dfrac{d u^n}{dx^n} + p_1 \dfrac{d^{n-1} u}{dx^{n-1}} + \dots \; p_n(x)\, u(x) = q(x)$

can always be written as a system of first-order ODEs $y_1' = y_2$, $y_2' = y_3$ etc. $\quad (u = y_1)$

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ -p_1 y_n - p_2 y_{n-1} \cdots - p_n y_1 + q \end{pmatrix}$$

Linear:

$$\frac{d\mathbf{y}}{dx} - A(x)\,\mathbf{y}(x) = \mathbf{q}(x) \quad \text{with a matrix } A$$

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homogeneous problem: $y' = A(x)\, y(x)$ (*)

$\hookrightarrow$ solutions form a vector space

A system $\{\varphi_i\}$ is called <u>fundamental system of solutions</u> if the φ_i are linearly independent.

Assume $\varphi_1(x) = \begin{pmatrix} \varphi_{11} \\ \vdots \\ \varphi_{n1} \end{pmatrix}$, $\varphi_2(x) = \begin{pmatrix} \varphi_{12} \\ \vdots \\ \varphi_{n2} \end{pmatrix}$, ...

$\Phi = (\varphi_1\ \varphi_2 \dots \varphi_n) = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{pmatrix}$ is called <u>fundamental matrix</u> and $W = \det(\Phi)$ is the <u>Wronskian</u>.

E.g.: $\Phi = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

for $y'' + y = 0$ $\quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

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- Liouville formula: $W(x) = W(x_0)\, e^{\int_{x_0}^{x} (\mathrm{tr}\, A(s))\, ds}$
- $W(x) \neq 0$: necessary and sufficient for $\{\varphi_k\}$ to be a fundamental system.

Particular solution:

$$y' - Ay = q$$

Assume $\{\varphi_k\}$ is a fundamental system

$$\varphi_p = c_1 \varphi_1 + c_2 \varphi_2 \ldots + c_n \varphi_n = \sum c_k \varphi_k$$

$$\sum c_k \varphi_k' + c_k' \varphi_k - c_k A \varphi_k = q$$

$$\sum \varphi_k c_k' = q \rightarrow \text{solve for } c_k$$

to obtain a particular solution.

Autonomous System:

$$y'(x) = F(y)$$ ← no explicit x-dependence

Linear: $y'(x) = Ay$

$$\Rightarrow \quad y(x) = e^{Ax} y_0 \quad \text{(matrix exponential)}$$

Note: This works as $\frac{d}{dx}\left(e^{Ax}\right) = A e^{Ax}$

for matrices. In general $\frac{d}{dx}\left(e^{A(x)}\right)$ is not

$$A' e^{A(x)}$$

$$(\text{unless } [A', A] = 0)$$

Critical point: Let $y' = F(y)$ and $F(c) = 0$

Then c is called critical point.

The behaviour close to a critical point is dominated by the corresponding linearization.

$n=2$: $\frac{dx}{dt} = ax + by \quad \frac{dy}{dt} = cx + dy \qquad (*)$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

Critical points: $ax + by = 0$, $cx + dy = 0$ $\Rightarrow$ 0 is the only critical point if $\det(A) \neq 0$.

Secular equation:

$(x(t), y(t))$ is a solution of $(*)$, then x and y solve

$$\Rightarrow \quad u'' - (a+d)u' + (ad - bc)u = 0$$

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Proof: $by = x' - ax$

$by' = x'' - ax'$

$\Rightarrow \quad x'' - ax' = b(cx + dy) = bcx + d(x' - ax)$

$x'' - (a+d)x' + (ad - bc)x = 0 \qquad \square$

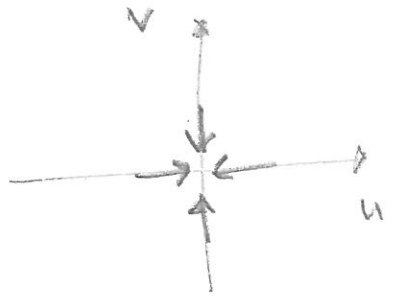
On the other hand: The characteristic polynomial of

A is $\lambda^2 - (a+d)\lambda + (ad - bc) = \det(A - \lambda I)$

$\Rightarrow$ The behaviour close to the critical point is determined by the eigenvalues λ_1 and λ_2 of A.

Example: $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

Clearly $\lambda_1 = -2$, $\lambda_2 = -3$: $\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$



(improper node)

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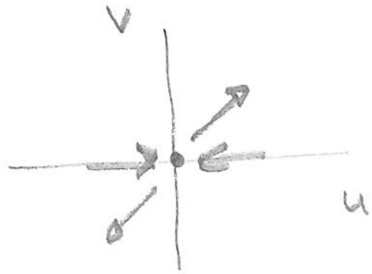
Saddle:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$\Rightarrow \lambda_{\pm} = -1, 2$, eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$$



Other possibilities: proper node (eigenvalues are the same and real)

spiral point (complex conjugates)

center (pure imaginary)

Limit cycle: nonlinear effect can be important!

$$x' = x + y - x(x^2+y^2) \qquad x = r\cos\theta$$

$$y' = -x + y - y(x^2+y^2) \qquad y = r\sin\theta$$

$$x' = r'\cos\theta - r\theta'\sin\theta$$
$$y' = r'\sin\theta + r\theta'\cos\theta$$

$$r'\cos\theta - r\theta'\sin\theta = r\cos\theta + r\sin\theta - r^3\cos\theta$$
$$r'\sin\theta + r\theta'\cos\theta = -r\cos\theta + r\sin\theta - r^3\sin\theta$$

$$\theta' = -1, \qquad r' = r(1-r^2)$$

trivial solution: $r = 1$, $\theta = -t + c$

$$r(t) = \frac{1}{\sqrt{1 + \left(\frac{1-r_0^2}{r_0^2}\right) e^{-2t}}}$$

$\Rightarrow$ trajectories approach the unit circle (limit cycle)