Math Methods

Lecture 3

Hilbert Spaces

(1)

Inner product space + completeness.

<u>Vector space</u>: $(V, +, \cdot)$

1. operation "+": $V \times V \to V$ $\quad x+y = y+x$
2. $\exists\, 0 \in V : \forall x \in V : \; x + 0 = x$
3. operation "$\cdot$": $K \times V \to V$

$$\alpha(\beta x) = (\alpha\beta)x \qquad 1(x) = x$$
$$\alpha(x+y) = \alpha x + \alpha y \qquad (\alpha+\beta)x = \alpha x + \beta x$$

K needs to be a Field, for us: $K = \mathbb{R}$ or $K = \mathbb{C}$.

<u>Examples</u>: $\mathbb{R}^n$, $\mathbb{C}^n$, set of all polynomials

<u>Inner product</u>: "$\cdot$": $V \times V \to K$

1. $(x,y) = (y,x)^*$
2. $(\alpha x + \beta y, z) = \alpha^*(x,z) + \beta^*(y,z)$
3. $(x,x) \geq 0$, $(x,x) = 0 \iff x = 0$

<u>Examples</u> 1. $\mathbb{C}^n$: $x = (\xi_1, \xi_2, \dots, \xi_n)$, $y = (\eta_1, \eta_2, \dots, \eta_n)$

$$(x,y) = \sum_{i=1}^{n} \xi_i^* \eta_i$$

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2. Assume f, g are polynomials on $[0,1]$:

$$(f,g) = \int_0^1 (f(x))^* g(x)\, w(x)\, dx$$

weight function

3. $x = (\xi_1, \xi_2, \xi_3, \xi_4)$, $y = (\eta_1, \eta_2, \eta_3, \eta_4)$ in $\mathbb{R}^4$

$$(x,y) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 - \xi_4\eta_4$$

not positive definite

Note that for a complex vector space: $(x,y) = (y,x)^*$.

$\Rightarrow$ We can define a length $\|x\| = \sqrt{(x,x)}$

(norm of the vector x)

Clearly $\|\alpha x\| = \sqrt{(\alpha x, \alpha x)} = \sqrt{|\alpha|^2 (x,x)} = |\alpha| \cdot \|x\|$.

$\Rightarrow$ we can define a metric $d(x,y) = \|x-y\|$ that measures the distance between points and analyze the geometry.

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Schwarz inequality: $|(x,y)| \leq \|x\|\|y\|$

Proof: $0 \leq (x+\alpha y, x+\alpha y) = (x,x) + \alpha(x,y) + \alpha^*(y,x)$
$+ |\alpha|^2 (y,y)$

Set $\alpha = \dfrac{-(x,y)^*}{(y,y)}$, multiply by (y,y):

$$0 \leq (x,x) - \frac{|(x,y)|^2}{(y,y)} - \frac{|(x,y)|^2}{(y,y)} + \frac{|(x,y)|^2}{(y,y)} \qquad \square$$

Note: equality if (and only if) x and y are linearly dependent.

"$\Rightarrow$": Assume $x = \alpha y$

$$\Rightarrow |(x,y)| = |\alpha|\,|(y,y)| = |\alpha|\,\|y\|^2 = \|\alpha y\|\|y\|$$
$$= \|x\|\|y\|$$

"$\Leftarrow$": Compute $\|(y,y)x - (y,x)y\|^2$ (and assume $(x,y)(x,y)^* = \|x\|^2\|y\|^2$)

$$= ((y,y)x - (y,x)y,\ (y,y)x - (y,x)y)$$

$$= \|y\|^4\|x\|^2 - \|y\|^2(y,x)(x,y) - (y,x)^*\|y\|^2(y,x)$$
$$+ (y,x)^*(y,x)\|y\|^2 = \|y\|^4\|x\|^2 - \|y\|^4\|x\|^2$$
$$- \|y\|^4\|x\|^2 + \|y\|^4\|x\|^2 = 0$$

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Triangle inequality:

$$\|x+y\| \le \|x\| + \|y\|$$

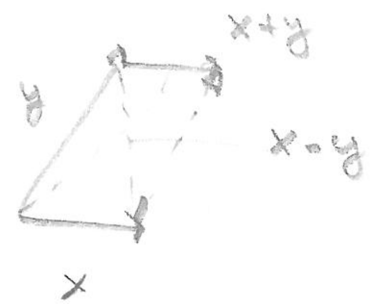
Proof:

$$\begin{aligned}\|x+y\|^2 &= (x,x) + (x,y) + (y,x) + (y,y) \\ &\le (x,x) + (y,y) + 2\,|(x,y)| \\ &\le \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2\end{aligned}$$

Parallelogram law: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Proof:

$$\|x+y\|^2 = \|x\|^2 + (x,y) + (y,x) + \|y\|^2$$
$$\|x-y\|^2 = \|x\|^2 - (x,y) - (y,x) + \|y\|^2$$



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Equivalence of norms over $\mathbb{C}$:

$\exists\ 0 < C_1 \leq C_2 : \ C_1 \|x\|_b \leq \|x\|_a \leq C_2 \|x\|_b$

1. We can consider $\|\cdot\|_b = \|\cdot\|_1$:

$$\|x\|_1 = \sum_{i=1}^{n} |\alpha_i| \quad \text{if} \quad x = \sum_{i=1}^{n} \alpha_i e_i$$

Assume both $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent to $\|\cdot\|_1$

$$C_1 \|x\|_1 \leq \|x\|_a \leq C_2 \|x\|_1$$
$$C_1' \|x_1\| \leq \|x\|_b \leq C_2' \|x\|_1$$

$$\Rightarrow \quad \frac{C_1'}{C_2} \|x\|_a \leq C_1' \|x_1\| \leq \|x\|_b \leq C_2' \|x_1\| \leq \frac{C_2'}{C_1} \|x\|_a$$

2. We can consider only x with $\|x\|_1 = 1$

Need to show $C_1 \|x\|_1 \leq \|x\|_a \leq C_2 \|x\|_1$

divide by $\|x\|_1 \Rightarrow \quad C_1 \leq \left\| \frac{x}{\|x\|_1} \right\|_a \leq C_2$, $u = \frac{x}{\|x\|_1}$

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(3) Any norm $\|\cdot\|_a$ is continuous under $\|\cdot\|_1$:

$\forall \epsilon > 0 \ \exists \delta : \|x - x'\|_1 < \delta \Rightarrow | \|x\|_a - \|x'\|_a | < \epsilon$

First: $| \|x\|_a - \|x'\|_a | < \|x - x'\|_a$

$$\Big(\|x\|_a - \|x'\|_a = \|x' + (x - x')\|_a - \|x'\|_a$$
$$\leq \|x'\|_a + \|x - x'\|_a - \|x'\|_a = \|x - x'\|_a \Big)$$

$$\|x - x'\|_a \leq \sum_{i=1}^{n} |a_i - a_i'| \cdot \|e_i\|_a \leq \|x - x'\|_1 \cdot \left(\max_i \|e_i\|_a\right)$$

choose $\delta = \dfrac{\epsilon}{\max_i \|e_i\|_a}$

(4) Extreme value theorem: A continuous function ($\|\cdot\|_a$) on a compact set (unit sphere $\{u, \|u\|_1 = 1\}$) takes maximum and minimum.

$$C_1 = \min_{\|u\|_1 = 1} \|u\|_a, \quad C_2 = \max_{\|u\|_1 = 1} \|u\|_a$$

$$\Rightarrow \quad C_1 \leq \|u\|_a \leq C_2 \quad \text{for } \|u\|_1 = 1$$

$\square$

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<u>Orthogonality</u>: x and y are <u>orthogonal</u>: $(x,y)=0$ $\quad (x \perp y)$

A set $\{x_1, x_2, \dots x_n\}$ is called <u>orthonormal</u> if $(x_i, x_j) = \delta_{ij}$

Vectors of an orthonormal set are linearly independent.

<u>Bessel inequality</u>: $\{x_1, x_2, \dots x_n\}$ orthonormal set, $x \in V$

set $r_i = (x_i, x)$. Then $\|x\|^2 \geq \sum_i |r_i|^2$,

$$x' = x - \sum r_i x_i \perp x_j$$

<u>Proof:</u>

$$\begin{aligned}(x', x') &= \left(x - \sum r_i x_i , x - \sum r_j x_j \right) \\ &= \|x\|^2 + \sum r_i^* r_j (x_i, x_j) \\ &\quad - \sum r_i^* (x_i, x) \\ &\quad - \sum r_j (x, x_j) \\ &= \|x\|^2 + \sum |r_i|^2 - \sum |r_i|^2 - \sum |r_i|^2 \\ &= \|x\|^2 - \sum |r_i|^2 \geq 0\end{aligned}$$

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$$(x_j, x') = (x_j, x) - \sum r_i (x_j, x_i) = (x_j, x) - r_j = 0 .$$

$\square$

Completeness :

A sequence (x_n) is called a Cauchy-sequence if
$\forall \epsilon > 0 \; \exists n_0 \in \mathbb{N} : \forall n, m \in \mathbb{N} : \|x_m - x_n\| < \epsilon$

A sequence (x_n) is said to converge to x $\left(\lim_{n\to\infty} x_n = x\right)$
if $\forall \epsilon > 0 \; \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} \;\; \|x_n - x\| < \epsilon$

If every Cauchy sequence in a space is convergent, we say the space is complete.

Note: $\mathbb{Q}$ is not complete.

E.g. $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$, $x_1 = 1$

converges to $\sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$.

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Note: $\mathbb{R}$ is not countable

Proof: Assume $[0,1]$ is countable, then list all elements

$$r_1 = 0.r_{11}r_{12}r_{13}\dots$$
$$r_2 = 0.r_{21}r_{22}r_{23}\dots$$
$$r_3 = 0.r_{31}r_{32}r_{33}$$

construct $\tilde{r}$ that is not in the list

(e.g. $r = 0.\tilde{r}_1\tilde{r}_2 r_3$ with $\tilde{r}_j = 1$ if $r_{jj} \neq 1$

$\tilde{r}_j = 2$ if $r_{jj} = 1$

$\square$

Hilbert Space: Inner product space which is complete.

Examples:

1. $\mathbb{R}^n$, $\mathbb{C}^n$
2. $x = (x_1, x_2, \dots)$ and $\sum_{i=1}^{\infty} |x_i|^2 < \infty$

   $(x,y) = \sum_{i=1}^{\infty} x_i^* y_i$. This space is ℓ^2.

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3. Set of square-integrable functions,

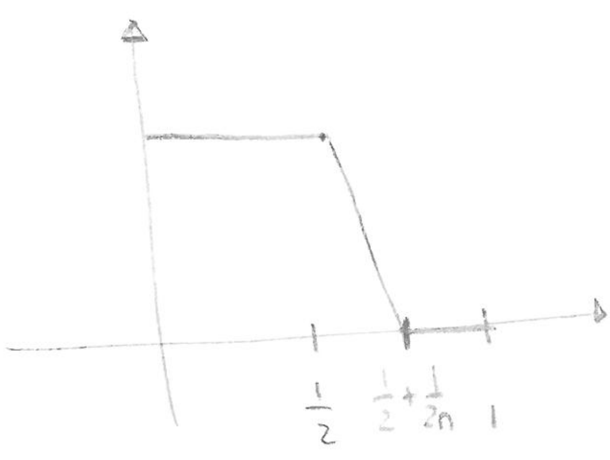
$L^2([a,b])$ or $L^2(\mathbb{R})$

$\int_a^b |f(x)|^2 dx < \infty$ or $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$

$$\langle f, g \rangle = \int_a^b f(x)^* g(x)\, dx$$

4. $C([0,1])$ space of continuous functions: incomplete (use $\|\cdot\|_2$)

$$f_n(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ 1 - 2n(x - \frac{1}{2}) & \frac{1}{2} \le x \le \frac{1}{2n} + \frac{1}{2} \\ 0 & \frac{1}{2n} + \frac{1}{2} \le x \le 1 \end{cases}$$



$f_n \to$ step-function

$$\|f_n(x) - f_m(x)\|_2 = \left(1 - \frac{n}{m}\right)\sqrt{\frac{1}{6n}} \to 0$$

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Banach Spaces:

Normed space: $\|\cdot\| : V \to \mathbb{R}$

1. $\|\lambda x\| = |\lambda| \, \|x\|$
2. $\|x+y\| \leq \|x\| + \|y\|$
3. $\|x\| = 0 \iff x = 0$

Example: 1. $\langle \, , \rangle$ induces a norm

2. On $\mathbb{R}^n$: $\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$

or $\|x\|_\infty = \max_i \{|x_i|\}$

Banach space: Complete normed space.

Ex: ℓ^p: $x = (x_1, x_2, \dots)$ with $\sum_{i=1}^{\infty} |x_i|^p < \infty$

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$$

L^p: f with $\int_a^b |f(x)|^p dx < \infty$

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$$

ℓ^p and L^p are Banach spaces.

Note: ℓ^2 and L^2 are Hilbert · Spaces.

For Hilbert spaces, we are interested in orthonormal basos: Assume $\{e_i\}$ are such an orthonormal basis, then we can write $x = \sum c_i e_i$ and obtain c_i via $(e_j, x) = \sum c_i (e_j, e_i) = c_j$

note: We can also have a basis that is not countable (see below: Fourier integral)