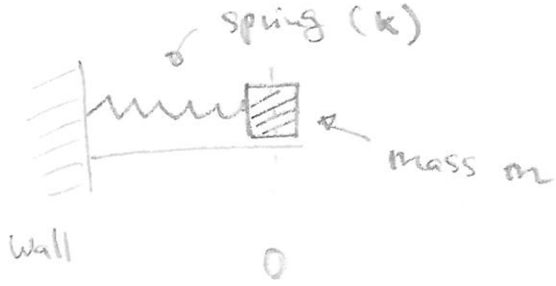
Math Methods

Lecture 1

The method of multiple scales:



x : deviation from rest state

Newton: $m\ddot{x} = m\dfrac{d^2x}{dt^2} = -kx$

$$\Rightarrow \quad \ddot{x} + \frac{k}{m}x = 0$$

linear ordinary differential equation (ODE)
2nd-order

Many ways to solve this equation:

(a) multiply by $\dot{x}$: $\quad \dot{x}\ddot{x} + \dfrac{k}{m}x\dot{x} = 0$

$$\Rightarrow \quad \frac{1}{2}\dot{x}^2 + \frac{1}{2}\frac{k}{m}x^2 = C \leftarrow \text{separable}$$

(b) Find a fundamental pair, e.g. $(\sin(\omega t), \cos(\omega t))$

$\left(\omega^2 = \dfrac{k}{m}\right) \quad \Rightarrow \quad x(t) = A\sin(\omega t) + B\cos(\omega t)$

A, B are found from initial conditions $(x(0), \dot{x}(0))$.

<u>Ex</u>: $x(0) = 0$, $\dot{x}(0) = 1 \Rightarrow A = 1$, $B = 0$

(c) ansatz: $x(t) = e^{\lambda t} \Rightarrow \lambda^2 e^{\lambda t} + \frac{k}{m} e^{\lambda t} = 0$

$$\Rightarrow \lambda^2 + \frac{k}{m} = 0 \Rightarrow \lambda = i\sqrt{\frac{k}{m}} \text{ or } \lambda = -i\sqrt{\frac{k}{m}}$$

$$\Rightarrow x(t) = \tilde{A} e^{i\omega t} + \tilde{B} e^{-i\omega t}$$

equivalent to (b) (Remember $e^{ia} = \cos a + i \sin a$)

<u>Friction</u>: add a (small) term $\boxed{\ddot{x} + \epsilon \dot{x} + x = 0}$

(assume $\frac{k}{m} = 1$ for simplicity)

Exact solution: $x(t) = e^{\lambda t} \Rightarrow$

$$\lambda^2 + \epsilon\lambda + 1 = 0$$

$$\lambda^2 + \epsilon\lambda + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{4} - 1$$

$$\left(\lambda + \frac{\epsilon}{2}\right)^2 = -\left(1 - \frac{\epsilon^2}{4}\right)$$

$$\lambda = -\frac{\epsilon}{2} + i\sqrt{1-\frac{\epsilon^2}{4}} \quad \text{or} \quad \lambda = -\frac{\epsilon}{2} - i\sqrt{1-\frac{\epsilon^2}{4}}$$

$$e^{-\frac{\epsilon}{2}t}\, e^{i\sqrt{1-\frac{\epsilon^2}{4}}\,t} \qquad e^{\frac{\epsilon}{2}t}\, e^{-i\sqrt{1-\epsilon^2/4}\,t}$$

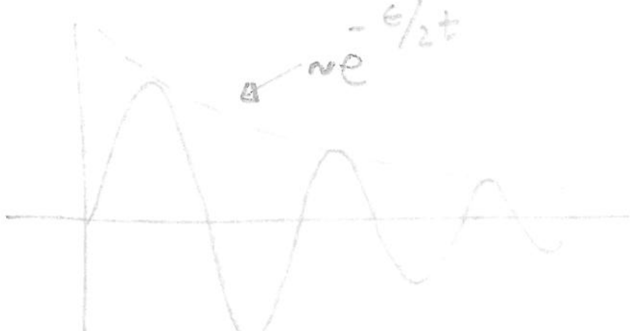
$$\Rightarrow \quad x(t) = \tilde{A}\, e^{-\frac{\epsilon}{2}t}\, e^{i\sqrt{1-\frac{\epsilon^2}{4}}\,t} + \tilde{B}\, e^{-\frac{\epsilon}{2}t}\, e^{-i\sqrt{1-\epsilon^2/4}\,t}$$

For $x(0)=0$, $\dot{x}(0)=1$:

$$x(t) = A\, e^{-\epsilon/2\, t} \sin\left(\sqrt{1-\frac{\epsilon^2}{4}}\,t\right) \quad \text{from } x(0)=0$$

$$\boxed{x(t) = \frac{1}{\sqrt{1-\epsilon^2/4}}\, e^{-\epsilon/2\, t} \sin\left(\sqrt{1-\frac{\epsilon^2}{4}}\,t\right)}$$

decaying exponential



very often: we are interested in an <u>approximation</u>.
Main tool: Taylor expansion.

<u>Here</u>: Taylor expansion is more difficult to apply than it might seem...

Naive expansion: $\quad x(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots$

Leading order: $\quad \ddot{y}_0 + y_0 = 0 \;, \quad y_0(0) = 0 \quad \dot{y}_0(0) = 1$

$\Rightarrow \quad y_0(t) = \sin t$

$\mathcal{O}(\epsilon)$: $\quad \ddot{y}_1 + y_1 = -\dot{y}_0 = -\cos t \qquad \begin{matrix} y_1(0) = 0 \\ \dot{y}_1(0) = 0 \end{matrix}$

$\Rightarrow \quad y_1(t) = -\frac{1}{2} t \sin t$ (check!)

Thus: $\quad x(t) \approx y_0(t) + \epsilon y_1(t) = \sin t - \frac{1}{2}\epsilon t \sin t$

only valid for very small t.

Multiple time scales:

Try $\quad x(t) = y_0(t_0, t_1) + \epsilon y_1(t_0, t_1) + \dots$

$t_0 = t \;, \quad t_1 = \epsilon t$

$$\Rightarrow \quad \frac{dy_0}{dt} = \frac{\partial y_0}{\partial t_0} + \epsilon \frac{\partial y_0}{\partial t_1} \;, \quad \frac{d^2 y_0}{dt^2} = \frac{\partial^2 y_0}{\partial t_0^2} + 2\epsilon \frac{\partial^2 y_0}{\partial t_0 \partial t_1} + \underbrace{\epsilon^2 \frac{\partial^2 y_0}{\partial t_1^2}}_{\mathcal{O}(\epsilon^2)}$$

Leading order: $$\frac{\partial^2 y_0}{\partial t_0^2} + y_0 = 0 \,, \quad y_0(0) = 0 \quad \left.\frac{\partial y_0}{\partial t_0}\right|_{t=0} = 1$$

$$\Rightarrow y_0(t) = A(t_1)\sin(t_0) + B(t_1)\cos(t_0)$$

$$\frac{\partial y_0}{\partial t_0} = A(t_1)\cos(t_0) - B(t_1)\sin(t_0)$$

$$\Rightarrow \quad A(0) = 1, \quad B(0) = 0$$

$\mathcal{O}(\epsilon)$: $$\epsilon\left(\frac{\partial^2 y_1}{\partial t_0^2} + y_1\right) = -\epsilon\frac{\partial y_0}{\partial t_0} - 2\epsilon\frac{\partial^2 y_0}{\partial t_0 \partial t_1} =: \epsilon R \quad (*)$$

Idea: use dependence on t_1 (slow scale) to "remove" resonances

$$R = -\left[A(t_1)\cos(t_0) - B(t_1)\sin(t_0)\right] - 2\left[A'(t_1)\cos(t_0) - B'(t_1)\sin(t_0)\right]$$

This can be done in several ways:

(a) "naive approach": Try $R = 0$ (here ok)

(b) "pedestrian approach": solve $(*)$ (lengthy)

(c) "use a trick": multiply by $\sin(t_0)$, $\cos(t_0)$ and integrate

(d) "correct method": Fredholm alternative

(a):

$$-A - 2A' = 0 \qquad B + 2B' = 0$$

$$A(t_1) = A(0)\, e^{-t_1/2} \qquad B(t_1) = B(0)\, e^{-t_1/2}$$

$$\Rightarrow \quad y_0(t_0, t_1) = A(0)\, e^{-t_1/2} \sin(t_0) + B(0)\, e^{-t_1/2} \cos(t_0)$$

$$\boxed{x(t) \approx e^{-\epsilon t/2} \sin t}$$ "good approximation"

(c): what happens if we multiply (*) by $\cos(t_0)$ and integrate over $[0, 2\pi]$

Recall: $$\frac{1}{2\pi}\int_0^{2\pi} \cos^2(t_0)\, dt_0 = \frac{1}{2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(t_0)\cos(t_0)\, dt_0 = 0$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} R(t_0,t_1)\cos(t_0)\,dt_0 = -\frac{1}{2}\left(A(t_1) + 2A'(t_1)\right)$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos(t_0)\left[\frac{\partial^2 y_1}{\partial t_0^2} + y_1\right] dt_0 = (**)$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos(t_0)\,\frac{\partial^2 y_1}{\partial t_0^2}\,dt_0 = \frac{1}{2}\cos(t_0)\frac{\partial y_1}{\partial t_0}\Bigg|_0^{2\pi}$$

u v'

$$+\frac{1}{2\pi}\int_0^{2\pi} \sin(t_0)\,\frac{\partial y_1}{\partial t_0}\,dt_0$$

$$= \frac{1}{2}\left[\frac{\partial y_1}{\partial t_0}\Bigg|_{t_0=2\pi} - \frac{\partial y_1}{\partial t_0}\Bigg|_{t_0=0}\right] - \frac{1}{2\pi}\int_0^{2\pi}\cos(t_0)\,y_1\,dt_0$$

If we assume y_1 to be 2π-periodic $\Rightarrow (**) = 0$

(d) Let A be a linear operator (e.g. a matrix)

IF $Af = g$ is solvable, then $\langle \varphi, g\rangle = 0$ for all $\varphi \in \mathrm{Ker}(A^{+})$

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Proof:

$$\langle \varphi, Af \rangle = \langle \varphi, g \rangle$$

$$\langle A^{+} \varphi, f \rangle = \langle \varphi, g \rangle$$

$$0 = \langle \varphi, g \rangle$$

Here: $g = R$, $A = \dfrac{\partial^2}{\partial t_0^2} + 1$, $\mathrm{Ker}(A)$ is given by $\varphi(t_0) = \alpha \cos(t_0) + \beta \sin(t_0)$