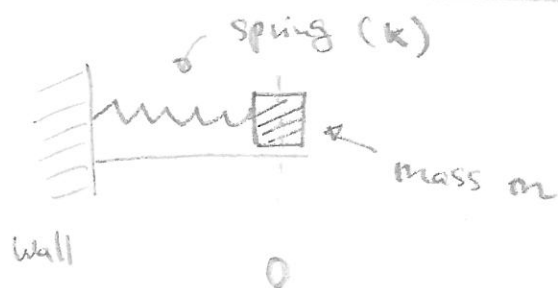


The method of multiple scales:

x : deviation from rest state

Newton: $m\ddot{x} = m \frac{d^2x}{dt^2} = -kx$

$$\Rightarrow \ddot{x} + \frac{k}{m}x = 0$$

linear ordinary differential equation (ODE)
2nd-order

Many ways to solve this equation:

(a) multiply by $\dot{x}$: $\dot{x}\ddot{x} + \frac{k}{m}x\dot{x} = 0$

$$\Rightarrow \frac{1}{2}\dot{x}^2 + \frac{1}{2}\frac{k}{m}x^2 = E \leftarrow \text{separable}$$

(b) find a fundamental pair, e.g. $(\sin(\omega t), \cos(\omega t))$

$$(\omega^2 = \frac{k}{m}) \Rightarrow x(t) = A \sin(\omega t) + B \cos(\omega t)$$

A, B are found from initial conditions $(x(0), \dot{x}(0))$.

Ex: $x(0) = 0, \dot{x}(0) = 1 \Rightarrow A = 1, B = 0$

(c) ansatz: $x(t) = e^{\lambda t} \Rightarrow \lambda^2 e^{\lambda t} + \frac{k}{m} e^{\lambda t} = 0$

$$\Rightarrow \lambda^2 = -\frac{k}{m} \Rightarrow \lambda = i\sqrt{\frac{k}{m}} \text{ or } \lambda = -i\sqrt{\frac{k}{m}}$$

$$\Rightarrow x(t) = \tilde{A} e^{i\omega t} + \tilde{B} e^{-i\omega t}$$

equivalent to (b) (Remember $e^{ia} = \cos a + i \sin a$)

Friction: add a (small) term $\ddot{x} + \epsilon \dot{x} + x = 0$

(assume $\frac{k}{m} = 1$ for simplicity)

Exact solution: $x(t) = e^{\lambda t} \Rightarrow$

$$\lambda^2 + \epsilon \lambda + 1 = 0$$

$$\lambda^2 + \epsilon \lambda + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{4} - 1$$

$$\left(\lambda + \frac{\epsilon}{2}\right)^2 = -\left(1 - \frac{\epsilon^2}{4}\right)$$

$$\lambda = -\frac{\epsilon}{2} + i\sqrt{1 - \frac{\epsilon^2}{4}} \text{ or } \lambda = -\frac{\epsilon}{2} - i\sqrt{1 - \frac{\epsilon^2}{4}}$$

$$\rightarrow e^{-\frac{\epsilon}{2}t} e^{i\sqrt{1 - \frac{\epsilon^2}{4}}t}$$

$$\rightarrow e^{-\frac{\epsilon}{2}t} e^{-i\sqrt{1 - \frac{\epsilon^2}{4}}t}$$

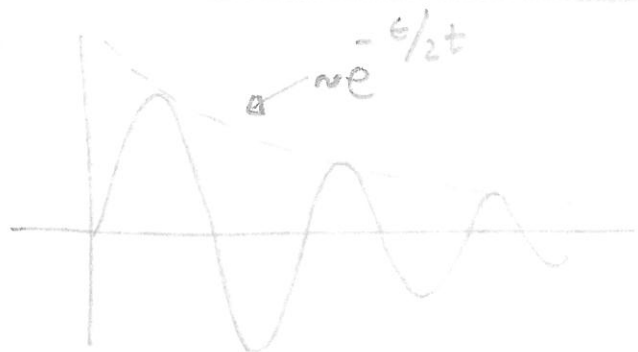
$$\Rightarrow x(t) = \tilde{A} e^{-\frac{\epsilon}{2}t} e^{i\sqrt{1-\frac{\epsilon^2}{4}}t} + \tilde{B} e^{-\frac{\epsilon}{2}t} e^{-i\sqrt{1-\frac{\epsilon^2}{4}}t}$$

For $x(0) = 0$, $\dot{x}(0) = 1$:

$$x(t) = A e^{-\frac{\epsilon}{2}t} \sin\left(\sqrt{1-\frac{\epsilon^2}{4}}t\right) \text{ from } x(0) = 0$$

$$x(t) = \frac{1}{\sqrt{1-\frac{\epsilon^2}{4}}} e^{-\frac{\epsilon}{2}t} \sin\left(\sqrt{1-\frac{\epsilon^2}{4}}t\right)$$

decaying
exponential



very often: we are interested in an approximation.
Main tool: Taylor expansion.

Here: Taylor expansion is more difficult to apply
than it might seem...

Naive expansion: $x(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots$

Leading order: $\ddot{y}_0 + y_0 = 0$, $y_0(0) = 0$, $\dot{y}_0(0) = 1$

$\Rightarrow y_0(t) = \sin t$

$\mathcal{O}(\epsilon)$: $\ddot{y}_1 + y_1 = -\dot{y}_0 = -\cos t$ $y_1(0) = 0$
 $\dot{y}_1(0) = 0$

$\Rightarrow y_1(t) = -\frac{1}{2} t \sin t$ (check!)

Thus: $x(t) \approx y_0(t) + \epsilon y_1(t) = \sin t - \frac{1}{2} \epsilon t \sin t$
 only valid for very small t .

Multiple time scales:

Try $x(t) = y_0(t_0, t_1) + \epsilon y_1(t_0, t_1) + \dots$

$t_0 = t$, $t_1 = \epsilon t$

$\Rightarrow \frac{d y_0}{d t} = \frac{\partial y_0}{\partial t_0} + \epsilon \frac{\partial y_0}{\partial t_1}$, $\frac{d^2 y_0}{d t^2} = \frac{\partial^2 y_0}{\partial t_0^2} + 2\epsilon \frac{\partial^2 y_0}{\partial t_0 \partial t_1} + \epsilon^2 \frac{\partial^2 y_0}{\partial t_1^2}$ $\mathcal{O}(\epsilon^2)$

Leading order: $\frac{\partial^2 y_0}{\partial t_0^2} + y_0 = 0$, $y_0(0) = 0$ $\left. \frac{\partial y_0}{\partial t_0} \right|_{t=0} = 1$

$$\Rightarrow y_0(t) = A(t_1) \sin(t_0) + B(t_1) \cos(t_0)$$

$$\frac{\partial y_0}{\partial t_0} = A(t_1) \cos(t_0) - B(t_1) \sin(t_0) \quad \left. \begin{array}{l} A(0) = 1 \\ B(0) = 0 \end{array} \right\}$$

$$O(\epsilon): \epsilon \left(\frac{\partial^2 y_1}{\partial t_0^2} + y_1 \right) = -\epsilon \frac{\partial y_0}{\partial t_0} - 2\epsilon \frac{\partial^2 y_0}{\partial t_0 \partial t_1} =: \epsilon R \quad (*)$$

Idea: use dependence on t_1 (slow scale) to "remove" resonances

$$R = - \left[A(t_1) \cos(t_0) - B(t_1) \sin(t_0) \right] - 2 \left[A'(t_1) \cos(t_0) - B'(t_1) \sin(t_0) \right]$$

This can be done in several ways:

(a) "naive approach": Try $R = 0$ (here ok)

(b) "pedestrian approach": solve (*) (lengthy)

- (c) "use a trick": multiply by $\sin(t_0)$, $\cos(t_0)$ and integrate
- (d) "correct method":

Fredholm alternative

(a):

$$-A - 2A' = 0 \quad B + 2B' = 0$$

$$A(t_1) = A(0) e^{-t_1/2} \quad B(t_1) = B(0) e^{-t_1/2}$$

$$\Rightarrow y_0(t_0, t_1) = A(0) e^{-t_1/2} \sin(t_0) + B(0) e^{-t_1/2} \cos(t_0)$$

$$x(t) \approx e^{-t/2} \sin t \quad \text{"good approximation"}$$

(c):

what happens if we multiply (*) by $\cos(t_0)$ and integrate over $[0, 2\pi]$

$$\text{Recall: } \frac{1}{2\pi} \int_0^{2\pi} \cos^2(t_0) dt_0 = \frac{1}{2}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(t_0) \cos(t_0) dt_0 = 0$$

$$\Rightarrow \frac{1}{2\pi} \int_0^{2\pi} R(t_0, t_1) \cos(t_0) dt_0 = -\frac{1}{2} (A(t_1) + 2A'(t_1))$$

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(t_0) \left[\frac{\partial^2 y_1}{\partial t_0^2} + y_1 \right] dt_0 = (**)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(t_0) \frac{\partial^2 y_1}{\partial t_0^2} dt_0 = \frac{1}{2} \cos(t_0) \frac{\partial y_1}{\partial t_0} \Big|_0^{2\pi} + \frac{1}{2\pi} \int_0^{2\pi} \sin(t_0) \frac{\partial y_1}{\partial t_0} dt_0$$

$$= \frac{1}{2} \left[\frac{\partial y_1}{\partial t_0} \Big|_{t_0=2\pi} - \frac{\partial y_1}{\partial t_0} \Big|_{t_0=0} \right] - \frac{1}{2\pi} \int_0^{2\pi} \cos(t_0) y_1 dt_0$$

If we assume y_1 to be 2π -periodic $\Rightarrow (**)=0$

(d) Let A be a linear operator (e.g. a matrix)

If $Af = g$ is solvable, then $\langle \varphi, g \rangle = 0$ for all $\varphi \in \text{Ker}(A^*)$

Proof:

$$\langle \varphi, Af \rangle = \langle \varphi, g \rangle$$

$$\langle A^* \varphi, f \rangle = \langle \varphi, g \rangle$$

$$0 = \langle \varphi, g \rangle$$

Here: $g = R$, $A = \frac{\partial^2}{\partial t_0^2} + 1$, $\text{Ker}(A)$ is given

by $\varphi(t_0) = \alpha \cos(t_0) + \beta \sin(t_0)$