Problem 1

Consider a stochastic differential equation of the form

$$dX_t = b(X_t)dt + \sigma dW, \qquad b(X_t) = -kX_t - \beta X_t^3$$

with $k > 0$ and $\beta > 0$. Show that $x = 0$ is a stable fixed point. Find the minimum action path (instanton) for the exit of from $x = 0$ in the limit $t \to \infty$.

Problem 2

Consider two stocks S_1 and S_2 whose evolution is given by geometric Brownian motion with volatilities σ_1 and σ_2 and drifts μ_1 and μ_2. Assume that the respective Brownian motions are correlated with a correlation coefficient ρ. Assume that there are no interest rates.

1. Use the Girsanov theorem to construct a risk-free measure $\mathbb{Q}$. Write S_1 and S_2 using the corresponding Brownian motions $\tilde{W}_1$ and $\tilde{W}_2$.
2. Consider a claim X that pays \$100 if at maturity T we have $S_1 > S_2$. Develop a pricing formula for such a claim. You can leave your answer as an integral.

Problem 3

Consider the following short rate model:

$$dr_t = \sigma dW_t + (\theta - \alpha r_t)dt$$

with constants α, θ, and σ.

1. Assume that at $t = 0$ we have the initial value r_0. Find mean and variance of r_t.
2. Find the corresponding $\sigma(t, T)$ in the HJM framework.