5 - Jumps

Poisson - distribution: $X \sim \mathrm{Poi}(\lambda)$: range $0 \dots \infty$

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\mathbb{E}(e^{sX}) = e^{\lambda(e^s - 1)}$$

Proof: $$\mathbb{E}(e^{sX}) = \sum_{k=0}^{\infty} e^{sk} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^s)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda e^s} = e^{\lambda(e^s - 1)}$$

Time - dependent: N_t counts number of events that happened in $[0,t]$. Now $\lambda \to \lambda t$, so $N_t \sim \mathrm{Poi}(\lambda t)$

Inhomogeneous: $\lambda = \lambda(t)$ as instantaneous rate

$N_t \sim \mathrm{Poi}(a(t))$ where $a(t) = \int_0^t \lambda_s \, ds$

compensated N_t: $\overline{N}_t = N_t - \lambda t$ is a martingale

$$\mathbb{E}(\overline{N}_t \mid \mathcal{F}_s) = \mathbb{E}(N_t - \lambda t \mid \mathcal{F}_s)$$

$$\varphi(s) = \mathbb{E}(e^{is\overline{N}_t}) = e^{\lambda t(e^{is} - 1 - is)}$$

$$= \mathbb{E}(N_s + (N_t - N_s) \mid \mathcal{F}_s) - \lambda(t + s - s)$$

$$= N_s - \lambda s + \underbrace{\mathbb{E}(N_t - N_s \mid \mathcal{F}_s) - \lambda(t-s)}_{=0}$$

5 - Jumps

2/6

__compounded process__: $N \sim Poi(\lambda)$, J_k is a sequence of independent random variables, distribution density f. Set $X = \sum_{k=1}^{N} J_k$ $\left(\mathbb{E}(e^{isJ_k}) = \int_{\mathbb{R}} e^{isx} f(x)\,dx \right)$

$$\mathbb{E}(e^{isX}) = \sum_{n \geq 0} \mathbb{E}(e^{isX} \mid N=n)\,\mathbb{P}(N=n)$$

$$= \sum_{n \geq 0} \mathbb{E}\left(e^{is\sum_{k=1}^{n} J_k}\right) e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= \sum_{n \geq 0} \left(\int_{\mathbb{R}} e^{isx} f(x)\,dx \right)^n e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= \exp\left(\lambda \int_{\mathbb{R}} (e^{isx} - 1) f(x)\,dx \right)$$

__Lévy-Jump-Diffusion__

$$N_t \sim Poi(\lambda t), \quad \mathbb{E}(J_k) = \varkappa$$

$$L_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - t\lambda\varkappa$$

5 - Jumps

Clearly L_t is a martingale $\iff b = 0$

$$\mathbb{E}\left(e^{isL_t}\right) = \exp\left\{t\left(isb - \frac{b^2\sigma^2}{2} + \int_{\mathbb{R}} (e^{isx} - 1 - isx)\,\lambda f(x)\,dx\right\}\right.$$

<u>α - stable distributions :</u>

$$\varphi_X(s) = e^{-c|s|^\alpha}$$ has "heavy tails" for $\alpha < 2$

$\alpha = 2 \longrightarrow$ normal distribution

$\alpha = 1 \longrightarrow$ Cauchy : $\mu \in \mathbb{R}$, $\sigma^2 > 0$

$$f(x) = \frac{\sigma}{\pi} \frac{1}{(x-\mu)^2 + \sigma^2}$$

$\alpha = \frac{1}{2} \longrightarrow$ Lévy ($\mu \in \mathbb{R}$, $\sigma^2 > 0$)

$$f(x) = \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} e^{-\sigma/2\,(x-\mu)}$$ for $x > \mu$

0 otherwise

5 - Jumps

General: Use FFT to invert.

$$\lim_{x\to\infty} \frac{P(|X|>x)}{x^{-\alpha}} = C_\alpha \qquad (*)$$

Algorithm:
- V uniform on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- W exponential, $\overline{W} = 1$
- $X_r = \dfrac{\sin(\alpha V)}{(\cos(V))^{1/\alpha}} \cdot \left(\dfrac{\cos(V-\alpha V)}{W}\right)^{(1-\alpha)/\alpha}$

↳ <u>Matlab</u>

- Show comparison:
  - random numbers
  - Fourier inverse

- Show Lévy processes (paths) for different parameters α.

- Proof of $(*)$: optional

5- Jumps

Option - Pricing : exponential Lévy models

B.S: $S_t = S_0 e^{rt + X_t}$ with $X_t = \sigma \tilde{W}_t - \frac{\sigma^2}{2} t$

such that $Z_t = B_t^{-1} S_t$ is a Q-martingale

Now: $S_t = S_0 e^{rt + X_t}$ where X_t is a Lévy process such that $Z_t = B_t^{-1} S_t$ is a martingale.

We assume that we know $\varphi(s)$, the char. function of X_t.

Call options : $C(K) = e^{-rT} \mathbb{E}\left((S_T - K)^+\right)$

(For simplicity: $S_0 = 1$) $c(k) = e^{-rT} \mathbb{E}\left(\left(e^{rT + X_T} - e^{k}\right)^+\right)$
$k = \ln K$)

idea: Obtain $c(k)$ using inverse Fourier transform (Carr & Madan)

5- Jumps

<u>technical difficulty:</u> $c(k)$ does not decay for $k \to -\infty$

trick: consider $z(k) = e^{-rT}\mathbb{E}\left[\left(e^{rT+X_T} - e^{k}\right)^+\right] - \left(1 - e^{k-rT}\right)^+$

<u>First:</u> $z(k) = e^{-rT}\int_{\mathbb{R}} p_T(x)\left(e^{rT+x} - e^{k}\right)\left(\mathbb{1}_{k \le x+rT} - \mathbb{1}_{k \le rT}\right)dx$

(exercise!)

Now we compute $\hat{z}(s) = \int_{\mathbb{R}} e^{isk}\, z(k)\, dk$

$$\hat{z}(s) = e^{-rT}\int_{-\infty}^{\infty} dk\, e^{isk}\int_{-\infty}^{\infty} dx\, p_T(x)\left(e^{rT+x} - e^{k}\right)\left(\mathbb{1}_{k \le x+rT} - \mathbb{1}_{k \le rT}\right)$$

$$= e^{-rT}\int_{-\infty}^{\infty} p_T(x)\,dx \int_{rT}^{x+rT} dk\, e^{isk}\left(e^{rT+x} - e^{k}\right)$$

$$= [\dots] = e^{isrT}\,\frac{\varphi(s-i) - 1}{is(1+is)}$$

(exercise!) and we find

$$z(k) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-isk}\, \hat{z}(s)\, ds$$

Tails of α-stable distributions

We show $f = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-isx} e^{-t|s|^\alpha} ds$ for $1<\alpha<2$

$$\approx \frac{\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)}{\pi} \frac{t}{x^{\alpha+1}} \quad \text{for } x \geq 0$$

Proof: $f = \frac{1}{2\pi}\left(\int_0^\infty e^{-isx} e^{-t|s|^\alpha} ds + \int_{-\infty}^0 e^{-isx} e^{-t|s|^\alpha} ds\right)$

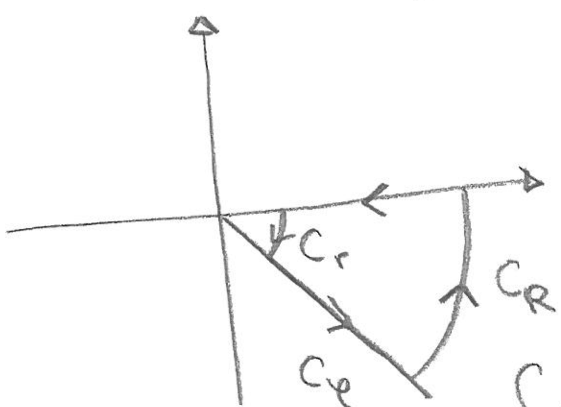
$$= \frac{1}{\pi}\int_0^\infty \cos(sx)\, e^{-ts^\alpha} ds$$

$$= \frac{1}{\pi} \mathrm{Re} \int_0^\infty e^{-isx} e^{-ts^\alpha} ds$$

Now set $g(z) = e^{-iz-z^\alpha}$, $z = \varrho e^{i\varphi}$

$e^{-z^\alpha} = e^{-\varrho^\alpha e^{i\varphi\alpha}}$ and $e^{i\varphi\alpha} = \cos\alpha\varphi + i\sin\alpha\varphi$

Set $\varphi = -\frac{\pi}{2\alpha}$



$\int_\Gamma dz\, g(z) = 0$

$\hookrightarrow$ Substitution:

$$sx = \xi e^{i\varphi} \qquad x\,ds = e^{i\varphi}\,d\xi \qquad \varphi = -\frac{\pi}{2\alpha}$$

$$J = \frac{1}{\pi x}\,\mathrm{Re}\left\{ e^{-i\frac{\pi}{2\alpha}} \int_0^\infty d\xi\; e^{-i\xi e^{i\varphi}}\; e^{-t\left(\frac{\xi}{x} e^{i\varphi}\right)^\alpha} \right\}$$

$$e^{-i\xi e^{i\varphi}} = e^{-i\xi e^{-\frac{i\pi}{2\alpha}}} = e^{-\xi e^{\frac{i\pi}{2} - \frac{i\pi}{2\alpha}}} = e^{-p\xi}$$

$$p = e^{\frac{i\pi}{2} - \frac{i\pi}{2\alpha}} = e^{i\pi\left(1-\frac{1}{\alpha}\right)/2}$$

$$e^{i\varphi\alpha} = e^{-i\frac{\pi}{2}} = -i$$

$$J = \frac{1}{\pi x}\,\mathrm{Re}\left\{ e^{-i\frac{\pi}{2\alpha}} \int_0^\infty d\xi\; e^{-p\xi + it(\xi/x)^\alpha} \right\}$$

Now $1 < \alpha < 2 \Rightarrow 1 > 1 - \frac{1}{\alpha} > 0 \Rightarrow \mathrm{Re}(p) > 0$

we can expand:

$$e^{it(\xi/x)^\alpha} = \sum_{j=0}^\infty \frac{1}{j!}\left(it\left(\frac{\xi}{x}\right)^\alpha\right)^j$$

$$= \sum_{j=0}^\infty \frac{1}{j!}\, e^{i\frac{\pi}{2}j} \left(\frac{t}{x^\alpha}\right)^j \xi^{j\alpha}$$

$$\mathcal{J} = \frac{1}{\pi x} \operatorname{Re}\left\{ \sum_{j=0}^{\infty} \frac{1}{j!} e^{i\pi(j-\frac{1}{\alpha})/2} \left(\frac{t}{x^\alpha}\right)^j \int_0^\infty d\xi\, e^{-p\xi} \xi^{j\alpha} \right\}$$

$$\int_0^\infty d\xi\, \xi^{j\alpha} e^{-p\xi} = \frac{1}{p} \int_0^\infty d\tau\, e^{-\tau} \frac{\tau^{j\alpha}}{p^{j\alpha}}$$

$$(\tau = p\xi)$$

$$= \frac{1}{p^{j\alpha+1}} \Gamma(1+j\alpha)$$

$$p^{j\alpha+1} = e^{i\pi(j\alpha+1)(1-\frac{1}{\alpha})\frac{1}{2}} \Rightarrow$$

$$\beta = \frac{e^{i\pi(j-\frac{1}{\alpha})/2}}{p^{j\alpha+1}} = e^{i\pi j/2} e^{-\frac{i\pi}{2\alpha}} e^{-i\pi(j\alpha - j + 1 - \frac{1}{\alpha})\frac{1}{2}}$$

$$= e^{\frac{1}{2} i\pi(2j - j\alpha - 1)}$$

$$\operatorname{Re}\beta = \cos\left(\frac{2j-j\alpha-1}{2}\pi\right) = \sin\left(\frac{2j-j\alpha}{2}\pi\right) \qquad j \geq 1$$

$$= \sin\left(j\pi - \frac{j\alpha\pi}{2}\right) = (-1)^{j-1} \sin\left(\frac{j\alpha\pi}{2}\right)$$

Tails of α-stable distributions

$$f \approx \frac{1}{\pi x} \, \frac{t}{x^{\alpha}} \, \Gamma(1+\alpha) \sin\left(\frac{\pi\alpha}{2}\right)$$

5/24
Final
12.45

Introduction to Random Matrix Theory

1/2

Random Matrix → matrix with random entries.

Basic question: What can we say about the distribution of eigenvalues?

Simple example: Wigner matrices:

X symmetric, $X_{ij} = \frac{1}{\sqrt{N}} Z_{ij}$

and $\mathbb{E}(Z_{ij}) = 0$, $\mathbb{E}(Z_{ij}^2) = 1$ → simple MATLAB

$\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}$: Wigner's semicircle law

$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta(x-\lambda_i)$ is the empirical measure

(λ_i are eigenvalues of X).

Theorem: For any bounded function f:

$$\mathbb{P}\left(|\langle L_N, f\rangle - \langle \sigma, f\rangle| > \epsilon\right) \to 0$$

as $N \to \infty$.

(here $\langle \mu, f \rangle = \int \mu(x) f(x)\,dx$)

So we say that $L_N \to \sigma$ weakly, in probability.

Now $\overline{L_N}$ is the expected value of the random measure.

One can show that $\overline{L_N} \to \sigma$ in moments as $N \to \infty$: For any k

$$\langle \overline{L_N}, x^k \rangle \longrightarrow \langle \sigma, x^k \rangle = \begin{cases} 0 & k \text{ odd} \\ \underbrace{\frac{1}{\frac{k}{2}+1}\binom{k}{k/2}}_{\text{Catalan numbers}} & k \text{ even} \end{cases} \quad (*)$$

Idea of proving $(*)$:

$$\langle \overline{L_N}, x^k \rangle = \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N} \lambda_i^k\right)$$

$$= \frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\left(X^k\right)\right) = \frac{1}{N}\sum_{i=(i_1, i_2, \dots, i_k)} \mathbb{E}\left(X_{i_1 i_2} X_{i_2 i_3} \cdots X_{i_k i_1}\right)$$

Proof of semi-circle law

$$g_\mu(z) = \int_{\mathbb{R}} \frac{1}{x-z} \varrho_\mu(x)\,dx$$

if ϱ_μ is the density of μ. If all moments are finite:

$$g_\mu(z) = \int_{\mathbb{R}} -\frac{1}{z} \frac{1}{1-\frac{x}{z}} \varrho_\mu(x)\,dx = -\frac{1}{z} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \frac{x^k}{z^k} \varrho_\mu(x)\,dx$$

$$= -\frac{1}{z} \sum_{k=0}^{\infty} \frac{m_k}{z^k} \quad \text{with} \quad m_k = \int_{\mathbb{R}} x^k \varrho_\mu(x)\,dx$$

is the k-th moment of the measure μ.

Now $\mu = L_N$, where $L_N = \frac{1}{N} \sum_{i=1}^{N} \delta(x-\lambda_i)$

is the empirical measure of eigenvalues

$$g_{L_N}(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - z} = \frac{1}{N} \operatorname{Tr}\left(\frac{1}{X_N - z\mathbb{1}}\right)$$

(In general we have $\sum_{i=1}^{N} f(\lambda_i) = \operatorname{Tr}(f(X_N))$)

(exercise)

Proof of semi-circle law

$$\mu\,[a,b] = \lim_{\eta \to 0} \int_a^b \frac{1}{\pi} \operatorname{Im} g_\mu(x+i\eta)\,dx$$

if $\mu\{a\} = \mu\{b\} = 0$.

<u>Proof:</u> $$\int_a^b \frac{1}{\pi} \operatorname{Im} g_\mu(x+i\eta)\,dx = J(a,b)$$

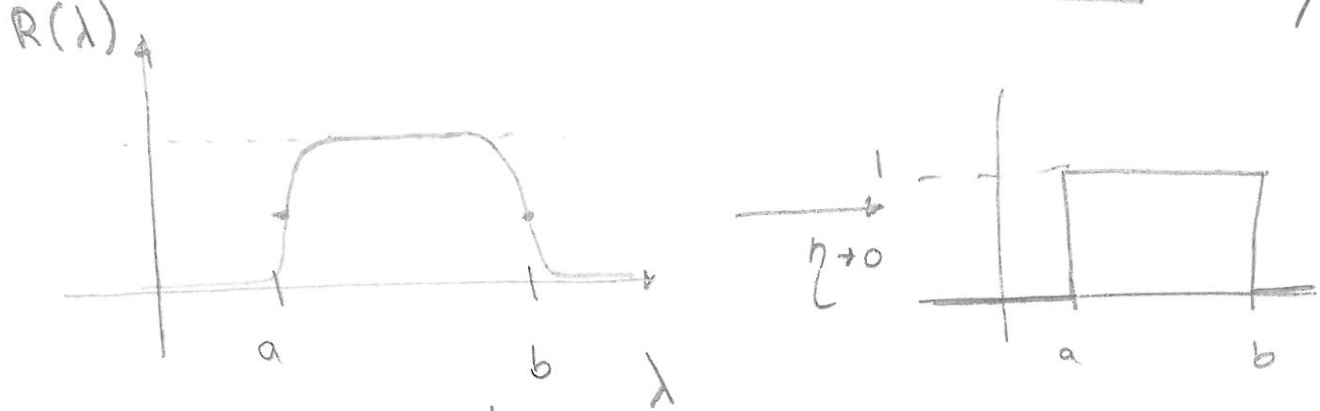
$$= \int_a^b \frac{1}{\pi} \operatorname{Im} \int_{\mathbb{R}} \frac{1}{\lambda - x - i\eta}\, \rho_\mu(\lambda)\,d\lambda\,dx$$

$$= \int_a^b \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta}{(\lambda-x)^2+\eta^2}\, \rho_\mu(\lambda)\,d\lambda\,dx$$, carrying out integration over x

$$= \int_{-\infty}^{\infty} \frac{1}{\pi}\left(\tan^{-1}\left(\frac{b-\lambda}{\eta}\right) - \tan^{-1}\left(\frac{a-\lambda}{\eta}\right)\right) \rho_\mu(\lambda)\,d\lambda$$

$$R(\lambda) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{b-\lambda}{\eta}\right) - \tan^{-1}\left(\frac{a-\lambda}{\eta}\right)\right]$$

Proof of the semi-circle law



$$\Rightarrow \mathcal{J}(a,b) \to \int_a^b \rho_\mu(\lambda)\, d\lambda = \mu([a,b])$$

for $\eta \to 0$ as claimed $\square$

Wigner ensemble: Resolvent of X: $G_X(z) = (X - z)^{-1}$

Now X_N with $\mathbb{E}(X_{ij}) = 0$, $\mathbb{E}(Z_{ij}^2) = 1$ $\quad Z_{ij} = \sqrt{N} X_{ij}$

and X_N symmetric

$$g_{L_N}(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - z} = \frac{1}{N} \operatorname{Tr} G_{X_N}(z)$$

$$\mathbb{E}(g_{L_N}(z)) = \frac{1}{N} \mathbb{E}(\operatorname{Tr} G_{X_N}(z))$$

Proof of the semi-circle law

$$G_{X+A}(z) - G_X(z) = - G_{X+A}(z)\, A\, G_X(z) \qquad (*)$$

$$(X+A-z)^{-1} - (X-z)^{-1} = -(X+A-z)^{-1} A (X-z)^{-1}$$

$$\left((X+A-z)^{-1} - (X-z)^{-1}\right)(X-z) = -(X+A-z)^{-1} A$$

$$(X+A-z)(X-z) - \mathbb{1} = -(X+A-z)^{-1} A$$

$$X-z-(X+A-z) = -A \qquad \checkmark$$

Set $A = -X$ in $(*)$:

$$-\frac{1}{z} - G_X(z) = \frac{-1}{z} X G_X(z) \quad (\bullet) \qquad \left(\frac{1}{z} \equiv \frac{1}{z}\mathbb{1}\right)$$

and
$$\frac{\partial G_{uv}}{\partial X_{ij}} = - G_{ui} G_{jv} - G_{uj} G_{iv} \qquad \text{(exercise)}$$

$$(\bullet) \Rightarrow \quad G_X(z) = -\frac{1}{z} + \frac{1}{z} X G_X(z)$$

Proof of the semi-circle law

<u>Lemma:</u> If $\xi \sim \mathcal{N}(0, \sigma^2)$ and f is differentiable

$$\mathbb{E}\left(\xi f(\xi)\right) = \sigma^2 \, \mathbb{E}\left(f'(\xi)\right) \qquad \text{(exercise)}$$

<u>Then:</u>

$$\mathbb{E}\left(\frac{1}{N} \mathrm{Tr}\left(G_{X_N}(z)\right)\right) = -\frac{1}{z} + \frac{1}{Nz} \mathbb{E}\left(\mathrm{Tr}\, X G_X\right)$$

$$= -\frac{1}{z} + \frac{1}{Nz} \sum_{ij} \mathbb{E}\left(X_{ij} G_{ji}\right)$$

$$= -\frac{1}{z} + \frac{1}{N^2 z} \sum_{ij} \mathbb{E}\left(\frac{\partial G_{ji}}{\partial X_{ij}}\right)$$

$$= -\frac{1}{z} + \frac{1}{N^2 z} \sum_{ij} \mathbb{E}\left(-G_{ji} G_{ij} - G_{jj} G_{ii}\right)$$

$$= -\frac{1}{z} \underbrace{- \frac{1}{N^2 z} \mathbb{E}\left(\mathrm{Tr}\left(G^2\right)\right)}_{A} - \frac{1}{z} \mathbb{E}\left(\left(\frac{1}{N} \mathrm{Tr}\, G\right)^2\right)$$

Set $g(z) = \frac{1}{N} \mathrm{Tr}\left(G_{X_N}(z)\right)$ we have

$$\mathbb{E}\left(g(z)\right) = -\frac{1}{z} - \frac{1}{z} \mathbb{E}\left(g(z)^2\right) + E_N$$

One can show that $\mathrm{Var}\, g(z) \to 0$ as $N \to \infty$

$\Rightarrow$ $\mathbb{E}(g(z)) = -\frac{1}{z} - \frac{1}{z}\left[\mathbb{E}(g(z))\right]^2$ converges to the solution of $s(z) = -\frac{1}{z} - \frac{1}{z} s(z)^2$ which is the Stieltjes transform of the semi-circle law (exercise)

Note: one can show that $A \to 0$ for $N \to \infty$.

$\boxed{\text{HW5}}$

(1) (10 points) Find the characteristic function $\varphi(s) = \mathbb{E}(e^{isX})$ if X has a Cauchy distribution $p(x) = \frac{\sigma}{\pi} \frac{1}{(x-\mu)^2 + \sigma^2}$

(2) (10 points) Consider as in the lecture an exponential Lévy model and

$$z(k) = e^{-rT} \mathbb{E}\left[(e^{rT+X_T} - e^k)^+\right] - (1 - e^{k-rT})^+$$

Show:

(a) $$z(k) = e^{-rT} \int_{\mathbb{R}} p_T(x) (e^{rT+x} - e^k)(\mathbb{1}_{k \leq x+rT} - \mathbb{1}_{k \leq rT})\, dx$$

(b) $$\hat{z}(s) = e^{isrT} \frac{\varphi(s-i) - 1}{is(1+is)}$$

where $z(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-isk} \hat{z}(s)\, ds$ and

$$\hat{z}(s) = \int_{\mathbb{R}} e^{isk} z(k)\, dk$$

HW 5

(10 points)

3 (a) Assume that X is a symmetric matrix and f is analytic. Show $\sum_{i=1}^{n} f(\lambda_i) = \mathrm{Tr}\,(f(X))$ if $\lambda_1 \dots \lambda_n$ are the eigenvalues of X.

(b) Let X be a Wigner matrix and $G_X(z)$ its resolvent. Show

$$\frac{\partial G_{uv}}{\partial X_{ij}} = -G_{ui}G_{jv} - G_{uj}G_{iv}$$

(c) Show: If $\xi \sim \mathcal{N}(0, \sigma^2)$ and f is differentiable we have $\mathbb{E}(\xi f(\xi)) = \sigma^2\, \mathbb{E}(f'(\xi))$.

f cannot grow to fast - can you make this statement more precise?

4 (10 points) Show that $s(z) = -\frac{1}{z} - \frac{1}{z}\, s(z)^2$ if $s(z)$ is the Stieltjes transform of the semi-circle law

$$\sigma(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$