Lecture 4: Bigger Models

(a) Quantos

Basic Model

A *quanto* is a contract that pays out in a currency that is different from the currency that the underlying asset is valued. This could be, for example, and option on a German stock, traded on the German market in Euros, but the pay off of the option is in USD. In order to price these contracts correctly, we need to adjust for the exchange rate that, by itself, needs to be modeled as a stochastic process that, in addition, might be correlated to the stock process. For our model, the following *four* quantities will play a role:

1. The stock price, in EUR, modeled as exponential Brownian motion:

$$S_t = S_0 \mathrm{e}^{\sigma_1 W_{1t} + \mu t}\,.$$

2. The exchange rate (the value of 1EUR in USD)

$$C_t = C_0 \mathrm{e}^{\rho\sigma_2 W_{1t} + \bar{\rho}\sigma_2 W_{2t} + \nu t}\,, \qquad \bar{\rho} = \sqrt{1-\rho^2}$$

3. The dollar cash bond (we can set, as usual, $B_0 = 1$) given by

$$B_t = B_0 \mathrm{e}^{rt}\,.$$

   with a risk-free interest rate r.

4. The Euro cash bond (we can set, as usual, $D_0 = 1$) given by

$$D_t = D_0 \mathrm{e}^{ut}\,.$$

   with a risk-free interest rate u.

The slightly complicated structure of the exchange rate results from our intuition that the stock and the exchange rate might be correlated. For the two independent $\mathbb{P}$-Brownian motions W_1 and W_2 we have

$$\mathbb{E}_{\mathbb{P}}(W_{1t}(\rho W_{1t} + \bar{\rho} W_{2t})) = \rho t \tag{1}$$

and $\rho W_{1t} + \bar{\rho} W_{2t}$ is clearly itself a $\mathbb{P}$-Brownian motion (exercise!). Since the pay off of the option we would like to price is in USD, we need to apply the exchange rate C_t in order to convert assets in EUR to their corresponding value in USD. The resulting quantities can then be hedged appropriately in the US market and, as usual, we need to take into account the discount factor given by the USD bond. Hence we need to consider the two processes Y_t and Z_t defined by

$$\begin{aligned} Y_t &= B_t^{-1} C_t D_t = B_0^{-1} D_0 C_0 \mathrm{e}^{\rho\sigma_2 W_{1t} + \bar{\rho}\sigma_2 W_{2t} + (\nu+u-r)t}, \qquad (2) \\ Z_t &= B_t^{-1} C_t S_t = B_0^{-1} C_0 S_0 \mathrm{e}^{\sigma_1 W_{1t} + \rho\sigma_2 W_{1t} + \bar{\rho}\sigma_2 W_{2t} + (\nu+\mu-r)t} \qquad (3) \end{aligned}$$

Applying Ito to both processes, we find

$$\begin{aligned} dY_t &= Y_t \left(\rho\sigma_2 dW_{1t} + \bar{\rho}\sigma_2 dW_{2t} + \left(\nu + u - r + \frac{1}{2}\sigma_2^2 \right) dt \right), \\ dZ_t &= Z_t \left((\sigma_1 + \rho\sigma_2) dW_{1t} + \bar{\rho}\sigma_2 dW_{2t} \right. \\ &\quad \left. + \left(\mu + \nu - r + \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2 \right) dt \right) \end{aligned}$$

In order to change to a drift-free measure, we can use again the Girsanov theorem, just in a multi-dimensional setting. Let's see whether we can define drifts γ_{1t} and γ_{2t} and $\mathbb{Q}$-Brownian motions $\tilde{W}_{1t} = W_{1t} + \gamma_{1t}$ and $\tilde{W}_{2t} = W_{2t} + \gamma_{2t}$ such that, under the new measure $\mathbb{Q}$, the processes Y_t and Z_t are drift-free. Under the new measure $\mathbb{Q}$, we can write the stock price and the exchange rate as

$$\begin{aligned} S_t &= S_0 \, \mathrm{e}^{\sigma_1 \tilde{W}_{1t} + (u - \rho\sigma_1\sigma_2 - \sigma_1^2/2)t}, \qquad (4) \\ C_t &= C_0 \, \mathrm{e}^{\rho\sigma_2 \tilde{W}_{1t} + \bar{\rho}\sigma_2 \tilde{W}_{2t} + (r - u - \sigma_2^2/2)t} \qquad (5) \end{aligned}$$

Assume we would like to price a digital claim that pays a dollar if the stock $S_T > K$. This contract has the price

$$V_0 = \mathrm{e}^{-rT} \mathbb{Q}(S_T < K) \,. \qquad (6)$$

Introducing $F = S_0 \mathrm{e}^{uT}$ (the local forward price), and $F_Q = F\mathrm{e}^{-\rho\sigma_1\sigma_2 T}$, the quanto forward price, we can see easily that

$$V_0 = \mathrm{e}^{-rT} \Phi \left(\frac{\ln(F_Q/K) - \sigma_1^2 T/2}{\sigma_1 \sqrt{T}} \right) \qquad (7)$$

The details of the derivation of this formula are left as an exercise.

(b) Interest Rate Models and the HJM Framework

Interest Rates

We have met interest rates before: So far, we assumed that the *cash bond* B_t evolves as a simple exponential of the form

$$B_t = B_0\, \mathrm{e}^{rt}, \qquad B_0 = 1$$

with a constant interest rate r. In reality, the situation is far more complicated: First, interest rates are random (so we need to introduce a variable interest rate r_t) and, second, there are more complex assets related to the interest market than the basic cash bond. Let's start by introducing the *discount bond*, a contract, initiated at $t = 0$ to deliver \$1 at maturity T. Clearly, the larger T, the smaller the price of the discount bond is - after all it is better to have money sooner than later. But once this contract is initiated it can still be bought or sold, so its value changes in time t, until it reaches \$1 at $t = T$. Therefore, the price of the discount bond depends on two variables, we can write $P(t, T)$, knowing that $P(T, T) = 1$. While $P(t = 0, T)$ is usually a rather smooth, decreasing function, we expect P to vary randomly with respect to t. Moreover, the value of P needs to have an impact on the current interest rate r_t which determines the evolution of the cash bond.

We start with introducing several useful terms - some of them are equivalent to P, but still very useful. First assume for a moment again that the interest rates were constant. Then, clearly, we would have $P(t, T) = \exp(-r(T-t))$ or $r = -\ln(P(t, T))/(T-t)$. This leads to the definition of the *yield* $R(t, T)$, generalizing this relationship to non-constant rates:

$$R(t, T) = -\frac{\ln P(t, T)}{T - t} \tag{8}$$

Setting $T = t + \delta t$ and looking at the short term yield leads to the definition of the instantaneous interest rate (or *short rate*):

$$r_t = \lim_{\delta t \to 0} R(t, t + \delta t) = -\frac{\ln P(t, t + \delta t)}{\delta t} = -\frac{\partial}{\partial T} \ln P(t, t) \tag{9}$$

where we have used the fact that $P(t, t) = 1$. This definition can be extended to define the *forward rate* as

$$f(t, T) = -\frac{\partial}{\partial T} \ln P(t, T), \qquad P(t, T) = \mathrm{e}^{-\int_t^T f(t,u)\, du} \tag{10}$$

A short calculation shows that

$$f(t,T) = R(t,T) + (T-t)\frac{\partial R(t,T)}{\partial T}\,. \tag{11}$$

A simple model

As a first start, let's write down a model for the forward rate and then try to develop a framework to price derivatives. At time $t = 0$ we are given as an initial condition $f(0,T)$ and a simple model should capture at least random and deterministic changes of the forward rate. Thus we try

$$d_t f(t,T) = \sigma dW_t + \alpha(t,T)dt, \tag{12}$$

or, equivalently after direct integration

$$f(t,T) = f(0,T) + \sigma W_t + \int_0^t \alpha(s,T)\,ds\,. \tag{13}$$

Note that this also implies a model for the interest rate as $r_t = f(t,t)$. We find

$$r_t = f(0,t) + \sigma W_t + \int_0^t \alpha(s,t)\,ds\,. \tag{14}$$

As for the stock market, we need two assets to from a tradable and we choose the cash bond B_t and the discount bond $P(t,T)$ for a fixed T. Then, as before, we can form the discounted process

$$Z_t = B_t^{-1} P(t,T) \tag{15}$$

and use the Girsanov theorem to construct a risk-neutral measure. To do so, however, we need to compute the drift of dZ_t. This calculation is not difficult, but slightly technical: We first find B_t and $P(t,T)$ explicitly by integration, then we apply Ito to find dZ_t. Remember that the cash bond and short rate satisfy the equations

$$dB_t = r_t B_t\,dt, \qquad r_t = f(0,t) + \sigma W_t + \int_0^t \alpha(s,t)\,ds\,. \tag{16}$$

We can directly integrate the equation for r_t and find

$$\int_0^t r_s\,ds = \int_0^t f(0,u)\,du + \sigma\int_0^t W_s\,ds + \int_0^t\int_s^t \alpha(s,u)du ds \tag{17}$$

Please take a minute to check the last term - you need to show that

$$\int_0^t \int_0^u \alpha(s,u) ds du = \int_0^t \int_s^t \alpha(s,u) du ds \,.$$

Putting everything together (no Ito-complications here!) we find

$$B_t = \mathrm{e}^{\int_0^t f(0,u)\,du + \sigma \int_0^t W_s\,ds + \int_0^t \int_s^t \alpha(s,u) du ds} \tag{18}$$

In order to find an explicit representation for the discount bond, we need to calculate

$$\int_t^T f(t,u)\,du = \sigma(T-t)W_t + \int_t^T f(0,u)du + \int_0^t \int_t^T \alpha(s,u) du ds \tag{19}$$

and we obtain for $P(t,T)$

$$P(t,T) = \mathrm{e}^{-\left(\sigma(T-t)\,W_t + \int_t^T f(0,u)du + \int_0^t \int_t^T \alpha(s,u) du ds\right)} \,. \tag{20}$$

Putting everything together, this yields

$$Z_t = B_t^{-1} P(t,T) = \mathrm{e}^{h_t} \tag{21}$$

where

$$\begin{aligned} h_t &= -\int_0^t f(0,u)\,du - \sigma \int_0^t W_s\,ds - \int_0^t \int_s^t \alpha(s,u) du ds \\ &\quad -\sigma(T-t)W_t - \int_t^T f(0,u)du - \int_0^t \int_t^T \alpha(s,u) du ds \\ &= -\left(\sigma(T-t)W_t + \sigma \int_0^t W_s\,ds + \int_0^T f(0,u)du + \int_0^t \int_s^T \alpha(s,u) du ds\right) \,. \end{aligned}$$

Applying Ito to Z_t (using the product rule for the term tW_t), we find

$$\begin{aligned} dZ_t &= Z_t \left(-\sigma(T-t)dW_t - \left(\int_t^T \alpha(t,u)du\right) dt + \frac{1}{2}\sigma^2 (T-t)^2\,dt\right) \\ &= -\sigma(T-t) Z_t (dW_t + \gamma_t dt) \end{aligned}$$

where we have introduced the drift term

$$\gamma_t = -\frac{1}{2}\sigma(T-t) + \frac{1}{\sigma(T-t)} \int_t^T \alpha(t,u)\,du \tag{22}$$

From here, we can directly proceed to use the Girsanov theorem to construct a risk-neutral measure $\mathbb{Q}$.
There is one additional aspect that we need to take into account: In our calculation we chose an arbitrary, but fixed maturity T. Clearly we could have chosen any other $\tilde{T}$ and done the calculations for the corresponding discounted process $\tilde{Z}_t$. Both Z_t and $\tilde{Z}_t$ are tradables and the measure $\mathbb{Q}$ should not depend on the choice of T or $\tilde{T}$. Otherwise, arbitrage possibilities will arise. Thus, we need to have the drift γ_t be *independent of* T, hence our model needs to satisfy the condition

$$\frac{\partial \gamma_t}{\partial T} = 0 \tag{23}$$

If we multiply the expression for γ_t by $(T-t)$ and take the derivative we find

$$\gamma_t = -\sigma(T-t) + \frac{1}{\sigma}\alpha(t,T)$$

or, the same statement re-arranged nicely

$$\alpha(t,T) = \sigma^2(T-t) + \sigma\gamma_t \tag{24}$$

We can now go back to the model for the forward rate: Using the Girsanov theorem to find $d\tilde{W}_t = dW_t + \gamma_t dt$, we have

$$\begin{aligned} d_t f &= \sigma dW_t + \alpha(t,T)dt \\ &= \sigma dW_t + \left(\sigma^2(T-t) + \sigma\gamma_t\right) dt \\ &= \sigma d\tilde{W}_t + \sigma^2(T-t)\, dt\,. \end{aligned}$$

For the forward rate and the instantaneous rate, we find

$$f(t,T) = \sigma\tilde{W}_t + f(0,T) + \sigma^2(2T-t)t/2, \qquad r_t = \sigma\tilde{W}_t + f(0,t) + \frac{1}{2}\sigma^2 t^2\,.$$

The Heath-Jarrow-Morton Framework

The above restriction on the drift to avoid arbitrage is a typical feature of modeling the interest market. In a straightforward way, we can extend this analysis to more complicated models, where the forward rate is given by

$$d_t f(t,T) = \sigma(t,T)dW_t + \alpha(t,T)\, dt\,. \tag{25}$$

Again we find for the interest rate by direct integration

$$r_t = f(0,t) + \int_0^t \sigma(s,t)dW_s + \int_0^t \alpha(s,t)\, ds \tag{26}$$

and we can form the cash bond process B_t and the discount bond $P(t,T)$

$$B_t = \mathrm{e}^{\int_0^t r_s \, ds}, \qquad P(t,T) = \mathrm{e}^{-\int_t^T f(t,u)\, du}$$

to set up the discounted process $Z_t = B_t^{-1} P(t,T)$. Introducing the $\Sigma(t,T)$ as

$$\Sigma(t,T) = -\int_t^T \sigma(t,u)\, du \tag{27}$$

it is easy to show (exercise!) that the corresponding drift is now

$$\gamma_t = \frac{1}{2}\Sigma(t,T) - \frac{1}{\Sigma(t,T)} \int_t^T \alpha(t,u)\, du \,. \tag{28}$$

As restriction we obtain

$$\alpha(t,T) = \sigma(t,T)\left(\gamma_t - \Sigma(t,T)\right) \tag{29}$$

which implies for $f(t,T)$ under the risk-neutral measure

$$d_t f(t,T) = \sigma(t,T) d\tilde{W}_t - \sigma(t,T)\Sigma(t,T) dt \tag{30}$$

Pricing derivatives follows now the usual mechanics by applying the martingale representation theorem. Again, choosing a maturity T for the discount bond, we can price a claim X depending on $P(t,T)$ with a maturity $S < T$. As usual, we will form

$$E_t = \mathbb{E}_{\mathbb{Q}}\left(B_S^{-1} X | \mathcal{F}_t\right) \tag{31}$$

and hold ϕ_t units of the T-bond at time t and $\psi_t = E_t - \phi_t Z_t$ units of the cash bond. The value of the portfolio will be

$$V_t = B_t E_t = B_t \mathbb{E}_{\mathbb{Q}}\left(B_S^{-1} X | \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^S r_u \, du} X | \mathcal{F}_t\right) \tag{32}$$

This formula extends in a trivial way to the S-bond itself: Such a bond is a claim $X = 1$ with maturity S. Hence we have

$$P(t,S) = \mathbb{E}_{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^S r_u \, du} | \mathcal{F}_t\right) \tag{33}$$

Short Rate Models

In short rate models, one specifies the interest dynamics r_t using the risk-neutral measure $\mathbb{Q}$. These models fall into the HJM frame work, as we can always construct from the short rate model (given by r_t) a volatility surface $\sigma(t,T)$ such that (30) is satisfied. Assume that we are give a model for r_t of the form

$$dr_t = \rho(r_t,t)d\tilde{W}_t + \nu(r_t,t)dt \tag{34}$$

then we can recover $f(t,T)$ in the following way: We know that the price $P(t,T)$ of the discount bond is related to the forward rate by

$$\int_t^T f(t,u)du = -\ln P(t,T) = -\ln \mathbb{E}_{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_u\,du}|\mathcal{F}_t\right)\,. \tag{35}$$

We can think of the last expression in this equation as a function $g = g(x,t,T)$ evaluated at $x = r_t$, hence define

$$g(x,t,T) = -\ln \mathbb{E}_{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_u\,du}|r_t = x\right)\,, \qquad f(t,T) = \frac{\partial g}{\partial T}(r_t,t,T) \tag{36}$$

and this function depends on r_t. From the HJM framework we know that

$$d_t f(t,T) = \sigma(t,T)d\tilde{W}_t - \sigma(t,T)\Sigma(t,T)dt$$

and we can use Ito's lemma to find a second expression for $d_t f(t,T)$;

$$d_t f(t,T) = \frac{\partial^2 g}{\partial x \partial T}(\rho(r_t,t)d\tilde{W}_t + \nu(r_t,t)dt) + \frac{\partial^2 g}{\partial t \partial T}dt + \frac{1}{2}\frac{\partial^3 g}{\partial x^2 \partial T}\rho^2(r_t,t)dt \tag{37}$$

Comparing the volatilities in the two representations, we find that

$$\sigma(t,T) = \rho(r_t,t)\frac{\partial^2 g}{\partial x \partial T}(r_t,t,T) \tag{38}$$

$$\Sigma(t,T) = -\rho(r_t,t)\frac{\partial g}{\partial x}(r_t,t,T)\,. \tag{39}$$

In the second expression, we simply used the definition of Σ.
The simplest short rate model is the *Ho and Lee model* where we have constant volatility σ and a deterministic drift θ_t:

$$dr_t = \sigma dW_t + \theta_t dt\,. \tag{40}$$

We can use (36) in order to compute the function $g(x,t,T)$ (exercise!):

$$g(x,t,T) = x(T-t) - \frac{1}{6}\sigma^2(T-t)^3 + \int_t^T (T-s)\theta_s\,ds \tag{41}$$

Clearly, we in HJM terms

$$\sigma(t,T) = \sigma \frac{\partial^2 g}{\partial x \partial T} = \sigma \tag{42}$$

and $\Sigma(t,T) = -\sigma(T-t)$, hence a forward rate $f(t,T)$ given by

$$d_t f(t,T) = \sigma d\tilde{W}_t + \sigma^2 (T-t) dt \tag{43}$$

which is actually the simple model discussed earlier. We now briefly discuss other models:

1. **Vasicek/Hull-White**: This model allows the short rate's drift to depend on its current value

   $$dr_t = \sigma d\tilde{W}_t + (\theta - \alpha r_t) dt \tag{44}$$

   Clearly, r_t will follow an Ornstein-Uhlenbeck process. For large t we have an equilibrium distribution with mean θ/α and variance $\sigma^2/(2\alpha)$. Therefore, the process is likely to fluctuate around the mean - however this model still has the draw-back that negative interest rates are in principle possible.

2. **Cox-Ingersoll-Ross**: Here, the multiplicative noise pushes the interest rate away from zero (and the process is mean-reverting as well).

   $$dr_t = \sigma_t \sqrt{r_t} d\tilde{W}_t + (\theta_t - \alpha_t r_t) dt \tag{45}$$

3. **Black-Karasinski** Use Ornstein-Uhlenbeck, but then use the exponential to guarantee that the process is positive:

   $$dX_t = \sigma_t d\tilde{W}_t + (\theta_t - \alpha_t X_t) dt, \qquad r_t = \mathrm{e}^{X_t} \tag{46}$$

As a particular example, let us discuss the Cox-Ingersoll-Ross model more in detail. Below a MATLAB code to simulate paths of this model: An important feature of the model is that the noise is actually multiplicative and tends to zero as the interest rate goes to zero. It can be shown that this prevents the process from taking negative values (for $\theta_t \geq \sigma_t^2$).

```
function [t,R] = cir(nSteps,mPaths)
% [t,R] = cir(nSteps,mPaths)
%
% Cox-Ingersoll-Ross model
%
% dr = (theta-a*r)*dt + sig*sqrt(r)*dW

T = 10;
dt = T/nSteps;

a = 2; sig = 1; theta = 2;

t = zeros(1,nSteps+1);
R = zeros(mPaths,nSteps+1);
R(:,1) = 1.5*ones(mPaths,1);

for k=1:nSteps
    R(:,k+1) = R(:,k) + (theta - a*R(:,k))*dt ...
        + randn(mPaths,1).*sqrt(R(:,k))*sig*sqrt(dt);
    t(k+1) = t(k) + dt;
end

end
```

Exercises

1. (10 points) Prove the formulas for S_t and C_t:

$$
\begin{aligned}
S_t &= S_0\,\mathrm{e}^{\sigma_1 \tilde{W}_{1t} + (u - \rho\sigma_1\sigma_2 - \sigma_1^2/2)t}, \\
C_t &= C_0\,\mathrm{e}^{\rho\sigma_2 \tilde{W}_{1t} + \bar{\rho}\sigma_2 \tilde{W}_{2t} + (r - u - \sigma_2^2/2)t}
\end{aligned}
$$

2. (10 points) Prove the formula for the digital quanto:

$$V_0 = \mathrm{e}^{-rT}\Phi\left(\frac{\ln(F_Q/K) - \sigma_1^2 T/2}{\sigma_1\sqrt{T}}\right)$$

3. (10 points) Consider the general HJM framework and show that

   (a) the drift γ_t is given by

$$\gamma_t = \frac{1}{2}\Sigma(t,T) - \frac{1}{\Sigma(t,T)}\int_t^T \alpha(t,u)\,du\,.$$

   (b) independence of γ_t from T yields the condition

$$\alpha(t,T) = \sigma(t,T)\,(\gamma_t - \Sigma(t,T))$$

4. (10 points) Show explicitly that, for the Ho and Lee model, we have

$$g(x,t,T) = x(T-t) - \frac{1}{6}\sigma^2(T-t)^3 + \int_t^T (T-s)\theta_s\,ds$$