Lecture 1: Financial markets and arbitrage

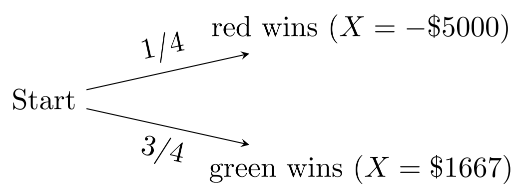
(a) Stocks, bonds, options, and arbitrage

The parable of the bookmaker

Consider a race between two horses (”red” and ”green”). Assume that the bookmaker estimates the chances of ”red” to win as 25% (and hence the chances of ”green” to win are 75%). This corresponds to 3-1 against ”red” (or 1-3 on ”green”). Let’s assume that \$5,000 are bet on ”red”, and \$10,000 on ”green”. We define a random variable X for the profit (or loss) of the bookmaker after the race. If ”red” wins, he needs to pay \$3·5,000, but keeps the \$10,000, so X is -\$5,000. If ”red” loses, ”green” wins, and the bookmaker has to pay \$10,000/3, but keeps the \$5,000.

So, in this case, X takes the value \$5,000/3 $\approx$ \$1667. In summary, the bookmaker might win or lose money. This means that there is a risk for the bookmaker - equivalent to himself was betting on the race.

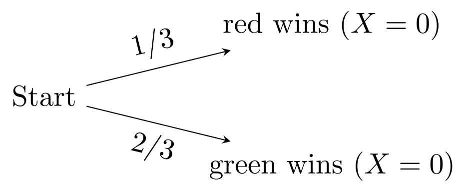
We can cast this in terms of probabilities: let $p = 1/4$ be the probability that ”red” wins. The diagram below illustrates the situation:



Clearly, we have $\mathbb{E}(X) = \$0$, but this is only an average taken over many (theoretical) realizations of X.

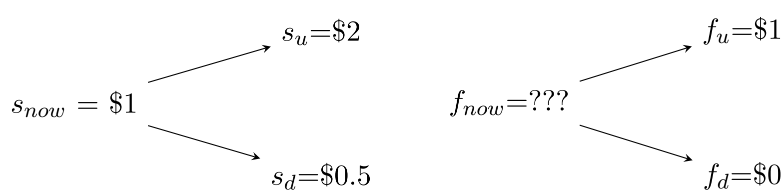
However, things do not have to be that way. The risk clearly depends on the way the bookmaker is quoting the odds. Therefore, we might ask the question: Is there a way to quote odds such that the bookmaker will remain *risk-neutral*? This might seem odd at first, but - from the point of view of the bookmaker, this is the most reasonable position to take. Of course, he will take a commission for his services and make a living in that way - without any risk related to the random outcome of the race.

Indeed, this is possible: If the bookmaker quotes the odds as 2-1 against ”red”, he will be risk-neutral: If ”red” wins, he needs to pay \$2·5,000, but keeps the \$10,000. If ”red” loses the race, he needs to pay \$10,000/2, but keeps the \$5,000. In either case, the bookmaker breaks even, there is no risk in selling the bets.



Note that these odds only depend on the sums of money that were bet on the horses - not on the real-world probabilities of the horses to win the race. In fact, such real-world probabilities are difficult to estimate, but in quoting the odds for the race they do not play any role for a bookmaker who intends to remain risk-neutral.

The situation is similar in finance when dealing with so-called *derivative* which are contract that are derived from fundamental assets. Consider, for instance a stock that is worth \$1. After a time δt, the stock can either go up to \$2 or go down to \$0.5. What is the price of a bet that pays \$1 if the stock goes up?



The main idea is that the seller of the bet can invest in the stock to hedge the claim and this possibility gives him a chance to sell the bet and still stay risk-neutral. All that he needs to do is to set up a portfolio that will have the worth of the claim after the time δt. Let's denote the value of the bet (after the time-tick) by $f_u = \$1$, if the stock goes up and $f_d = \$0$, if the stock goes down.

Consider a portfolio of ϕ units of stock and ψ units of a cash bond. For simplicity, we assume that the interest rate is zero. At the beginning, before the time-tick δt, the worth of the portfolio is

$$V = \phi S + \psi B\,.$$

Here, S is the current stock price (in our case \$ 1) and B - as we are working in dollars, we set $B = \$1$. When the clock ticks, the value of ψB will not change (since we assumed that the interest rate is zero), but the value of ϕS will change, since the value S after the time-tick δt is random. If the stock goes up, we will have $S = s_u = \$2$ and if the stock goes down, we will have $S = s_d = \$0.5$. If you are selling the bet and if you want to be risk-neutral, you will tri to adjust the portfolio (hence ϕ and ψ) such that V will have the value of \$1 if the stock goes up and \$0 if the stock goes down (to mimic the claim). It is easy to figure out what ϕ and ψ should be:

$$\begin{aligned} V_u &= \phi s_u + \psi B = f_u = 1 \\ V_d &= \phi s_d + \psi B = f_d = 0 \end{aligned}$$

This is an equation with two unknowns, and clearly we have

$$\phi = \frac{f_u - f_d}{s_u - s_d} = \frac{1-0}{2-0.5} = \frac{2}{3} \tag{1}$$

and, from either equation, we find $\psi = -1/3$. This means that, in oder to set up a risk-free portfolio that mimics the bet (claim), one needs

$$V = \frac{2}{3} \cdot \$1 - \frac{1}{3} \cdot \$1 = \$0.33\,.$$

And this is exactly the price (or worth) of the bet that the seller will ask from the buyer.

Basics of financial markets, derivatives

Stock and Bond: Our basic financial market consists of two types of assets: stocks and bonds. The stock is random, meaning that we cannot predict its value for future times. We will see later that exponential Brownian motion is a basic model and write

$$S_t = S_0 \, \mathrm{e}^{\mu t + \sigma W_t}\,.$$

The parameter μ is called the *average rate of return* and measures the average growth (or decline) of the stock. The parameter σ is called *volatility* and measures the strength of the price fluctuations. The precise meaning of W_t will become clear later - at this point we only need to know that S_t is random, hence that its future value is not known. The other asset, the cash bond, is deterministic. If we assume an interest $r \geq 0$ and compound continuously, we find that the value of the bond at a future time t is known:

$$B_t = B_0 \, \mathrm{e}^{rt} = B_0 \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt}$$

Most of the time, we will set $B_0 = 1$ (think of it as \$1 at time $t = 0$).
Basic assumptions: In our analysis, we usually assume the following:

- no transaction costs
- no tax
- unlimited borrowing/short-selling
- fixed interest rate, same rate if you borrow or lend
- all assets can be split
- no arbitrage

Derivatives: If an asset is derived from a basic asset, we call it a derivative. Options are important examples. The buyer of the option acquires the right (but has no obligation) to do something (usually to buy or to sell an asset for an agreed price) at a future time.

- *Call option:* gives the holder of the option the right to buy a stock for a price K.
- *Put option:* gives the holder of the option the right to sell a stock for a price K.

In both cases, we call K the *strike price.* The corresponding pay-offs are

- Pay-off of a call option: $(S_T - K)^+ = \max(S_T - K, 0)$.
- Pay-off of a put option: $(K - S_T)^+ = \max(K - S_T, 0)$.

Moreover, we distinguish between *European* and *American* options:

- *European option:* can only be exercised at the expiration date.
- *American option:* can be exercised at expiration date or any time before expiration date

In the following, for simplicity, we will focus on European options. Example: Consider a European call of a stock that is worth now $S_0 = \$100$ with strike price $K = \$120$ and a maturity of $T = 2$ years. If, at expiration, the stock is worth $S_T = \$150$, the worth of the call is the difference, hence \$30: The holder of the option will exercise the option, hence buy a stock for $K = \$120$ and then sell it at the current value of \$150. If, on the other hand, the stock happens to be worth $S_T = \$90$, the option will expire worthless (and not exercised, as nobody would pay $K = \$120$ for a stock that one can buy for \$90).

Arbitrage

Why is it so important to price options correctly? Consider, for example, a stock is worth \$100 now, a bond worth \$100 as well. Assume that the stock could go up or down \$20 in one year (so $s_u = \$120$ and $s_d = \$80$), and that the bond will be worth \$110. Assume that a bank offers a European call, $K = \$100$ for \$10. What would you do? Here is a smart idea: Buy 2/5 of the bond, one call option, sell 1/2 of the stock. The cost to set up this portfolio is

$$V = \frac{2}{5} \cdot 100 + 10 - \frac{1}{2} \cdot 100 = 0\,.$$

So, you can set up this portfolio for free. What will happen in one year? If the stock goes up, the call will be worth \$20 and, therefore,

$$V = \frac{2}{5} \cdot 110 + 20 - \frac{1}{2} \cdot 120 = 44 + 20 - 60 = 4\,.$$

If, on the other hand, the stock goes down, we find

$$V = \frac{2}{5} \cdot 110 + 0 - \frac{1}{2} \cdot 80 = 44 + 0 - 40 = 4\,.$$

We would have found a way to make money for free! Such arbitrage opportunities should not exist in a market that is in equilibrium - and a correct (risk-free inspired) pricing of options is essential for this.

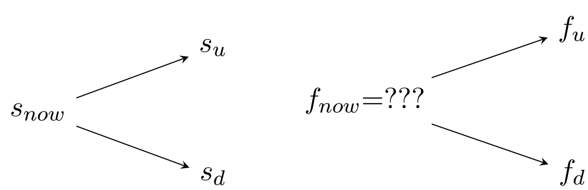
(b) Option pricing on a tree

Binomial pricing model with non-zero interest rates

Assume now that we have non-zero interest rates, such that ψ units of the cash bond B_0 will have the value

$$\psi B_0 \mathrm{e}^{r\delta t}$$

after a time δt. Again, we consider the basic binomial pricing situation where the stock can have two values after the time-tick δt, together with a derivative that also can have two values.



Again, we will set up a portfolio V at the beginning to replicate the claim, and the money we need to create this portfolio will correspond to the price that we charge for the derivative, hence $V_{now} = f_{now}$. Consider

$$V = \phi S + \psi B$$

Before the clock ticks, this portfolio has the value

$$V_{now} = f_{now} = \phi s_{now} + \psi B_0 \tag{2}$$

and, after the time-tick, we need to satisfy the equations

$$\begin{aligned} f_u &= \phi s_u + \psi B_0 \mathrm{e}^{r\delta t} \\ f_d &= \phi s_d + \psi B_0 \mathrm{e}^{r\delta t} \end{aligned}$$

and, again, we find that

$$\phi = \frac{f_u - f_d}{s_u - s_d}\,.$$

Note that this is the same formula as in the case of zero interest rates: In order to replicate the claim, a certain number of shares of the stock are necessary: The amount of stock to hold is given by the ratio of the difference of the claim and the difference of the stock given by the two possible scenarios. In the spirit of replicating the claim by the appropriate stock holding, we can also write equivalently

$$f_u - f_d = \phi(s_u - s_d)\,,$$

which is, therefore, trivial to remember. In order to proceed with the calculation of the initial price of the claim (or the initial portfolio), we still need to find the holding ψ of the cash bond. Going back to the above system of equations, if we multiply the first equation by s_d and the second equation by s_u and subtract the first equation from the second equation, we obtain directly

$$\psi = B_0^{-1}\mathrm{e}^{-r\delta t}\left(\frac{f_d s_u - f_u s_d}{s_u - s_d}\right) \tag{3}$$

and, putting everything together, we find the formula for the price of the initial portfolio (and hence the value of the claim before the time-tick) as

$$V = \frac{f_u - f_d}{s_u - s_d}\, s_{now} + \mathrm{e}^{-r\delta t}\left(\frac{f_d s_u - f_u s_d}{s_u - s_d}\right) \tag{4}$$

Again, it is important to see that this formula is entirely independent of any "real-world" probabilities that one might associate with the event of the stock S going up to s_u or going down to s_d. The price of the claim is determined by the idea of setting up a risk-less portfolio that replicates the claim, nothing else, a mechanism that is independent of the real-world probabilities. The estimate of the real-world probabilities might play a role, whether a buyer finds the claim attractive: Consider a call option that is attractive if the buyer believes that the stock is likely to go up: Even if most people feel, that there is a 90% chance of the stock to go up, the seller of the option will still price it independent of this probability: The option price is enforced by the requirement of setting up a risk-less portfolio.

Risk-neutral probability measure: The formula for the claim on a binomial branch (4) can be rewritten in a form that is much simpler to remember: First, define q as

$$q = \frac{\mathrm{e}^{r\delta t} s_{now} - s_d}{s_u - s_d}\,. \tag{5}$$

It can be shown that $0 < q < 1$. With this q, we can calculate the price of the initial portfolio as

$$V = \mathrm{e}^{-r\delta t}\left(q f_u + (1-q) f_d\right)\,. \tag{6}$$

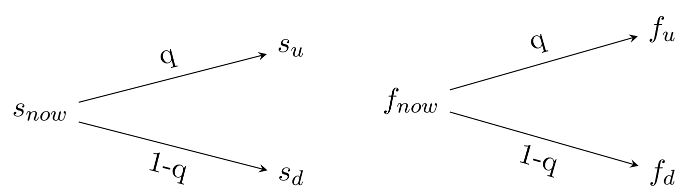
In the interest-free case $r = 0$, this formula has the form of an expectation for a binomial branch with the probability q.

$$V = f_{now} = qf_u + (1-q)f_d \tag{7}$$

Interestingly, in that case, the q is defined by (5) in a way that

$$s_{now} = qs_u + (1-q)s_d \tag{8}$$

In this way, the pricing formula is simple to remember (at least for the case $r = 0$): First find the *risk-neutral* probability q defined such that the current stock value is the expectation of two possible future stock values and then use this probability to compute the initial price of the claim by taking the expectation of the two possible claim values.



In this language, we can use all the tools from probability theory - knowing that q is not a "real-world" probability, but the risk-neutral probability. Returning to the example of a bet that pays \$1 if a stock goes up (and the stock, priced at \$1 now can take the values \$2 and \$0.50 after the time-tick), we find that

$$q = \frac{1-0.5}{2-0.5} = \frac{1}{3}, \qquad V = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 0 = \frac{1}{3}\,.$$

Example for a call option: As a second example, take the above case of the a stock that is worth now \$100 and can go up to \$150 or down to \$90. Consider a strike price $K = \$120$ and a time-tick of one year with $r = 5\%$. We then compute q as

$$q = \frac{\mathrm{e}^{r\delta t} s_{now} - s_d}{s_u - s_d} = \frac{100 \cdot \mathrm{e}^{0.05} - 90}{150 - 90} = 0.2521$$

and find as option value

$$V = \mathrm{e}^{-0.05}\,(q \cdot 30 + (1-q) \cdot 0) = 7.19$$

From branch to tree: With the basic pricing formula for a branch on hand, we can extend the pricing to trees easily. Simply apply the branch formula to each branch of the tree. Let's illustrate this idea using a simple example. Consider three time-ticks, and a stock with a value of 100 at the beginning that can go up or down 20 at each time-tick. Then, the corresponding stock process is given by the following tree:

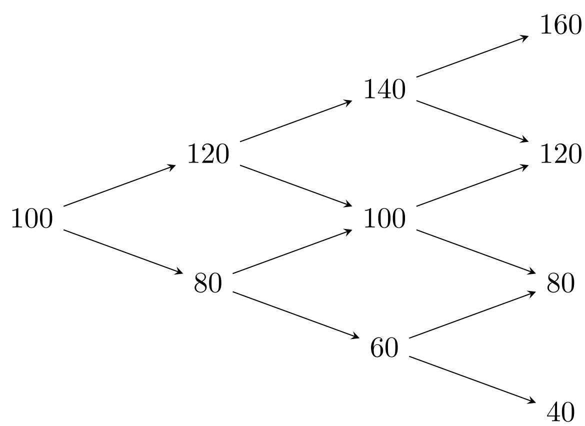


Figure 1: Binomial tree of the stock process

Consider, for instance, the branch on the top right corner. Here, $s_u = 160$, $s_d = 120$, and $s_{now} = 140$. Since the strike price of the option is $K = 100$, we know that, at the part of the tree, we have $f_u = 60$ and $f_d = 20$. From the stock values, we find immediately the risk-neutral probability q as

$$q = \frac{s_{now} - s_d}{s_u - s_d} = \frac{140 - 120}{160 - 120} = \frac{1}{2}$$

and, therefore, $f_{now} = qf_u + (1 - q)f_d = 40$. It is easy to see that, in this particular case, we have $q = 1/2$ on the entire tree. Since the option value is known at the end nodes of the tree, we can work backward and obtain the option price at the beginning of the tree (here 15). Figure shows the corresponding option tree. Working backward is a good idea as we need to set up a portfolio that replicates the value of the claim (here the option) at all nodes of the tree. Let us know see how the hedging works for a particular path:

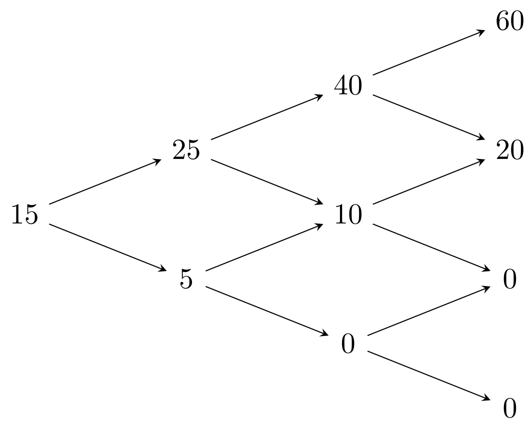


Figure 2: Binomial tree of the option process

Time $i = 0$ We are given 15 for the option. To set up the portfolio, we need to know how much of the stock is required. As before, we write the portfolio as

$$V = \phi S + \psi B$$

At time $i = 0$, we find the amount of stock we need to hold to be risk-neutral to be

$$\phi = \frac{f_u - f_d}{s_u - s_d} = \frac{25 - 5}{120 - 80} = \frac{1}{2}$$

As the price of the stock at the beginning of the tree is $S_0 = 100$, the cost is $0.5 \cdot 100 = 50$ and we are given 15 for writing the option. Hence we need to borrow 35 (and we assume, for simplicity, at the moment that the interest rate $r = 0$). Therefore, our bond holding is now -35. To summarize, at time $i = 0$, before the first time-tick, our holdings are $(\phi, \psi) = (0.5, -35)$.

Time $i = 1$ Assume that the stock goes up. Now we need to see whether we need to adjust our holdings of stock and bond. First, we need to compute the new ϕ. The ϕ necessary to be risk-neutral at this node of the tree is computed by looking at the next step ahead, hence

$$\phi = \frac{f_u - f_d}{s_u - s_d} = \frac{40 - 10}{140 - 100} = \frac{3}{4}$$

Since we are already holding 0.5 shares in our portfolio, we need to buy only 0.25 shares in addition. Now, the stock is worth $S_1 = 120$, hence the cost of the additional shares is $0.25 \cdot 120 = 30$, which brings our debt to 65. To summarize, at time $i = 1$, at the node where $S_1 = 120$, our holdings are $(\phi, \psi) = (0.75, -65)$.

Time $i = 2$ The new ϕ is now

$$\phi = \frac{f_u - f_d}{s_u - s_d} = \frac{60 - 20}{160 - 120} = 1$$

Note, that at this node, the option is in the money, for sure and will be exercised at the next time-tick. Considering our portfolio, we are already holding 3/4 of shares, so we need to buy 1/4 in addition. The price of the stock is now $S_2 = 140$, so the cost of this additional 0.25 is $0.25 \cdot 140 = 35$ which brings our debts to 100. To summarize, at time $i = 2$, at the node where $S_1 = 140$, our holdings are $(\phi, \psi) = (1, -100)$.

Time $i = 3$ The clock ticks one more time and the holder needs to decide whether to exercise the option. The bank, on the other hand, has created a portfolio whose value replicates the value of the claim (the option) such that the bank is risk-neutral. In our case, we see that if the stock goes up one more time, to $S_3 = 160$, the option in worth $S_3 - K = 60$. The portfolio has the same value:

$$V = \phi S + \psi B = 1 \cdot 160 - 100 \cdot 1 = 60$$

If, on the other hand, the stock happens to go down at the last step, hence $S_3 = 120$, the option is worth 20. Obviously, in this case, the portfolio has the value 20 as well:

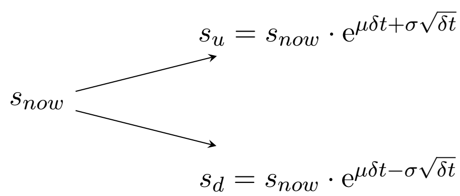
$$V = \phi S + \psi B = 1 \cdot 120 - 100 \cdot 1 = 20$$

Either way, the portfolio has the same value as the option at maturity, hence the bank remains risk-neutral. Note that, in order to maintain risk-neutrality, the bank had to adjust the holdings (ϕ, ψ) in the portfolio according to the stock movements. This is called a *dynamic hedging strategy*. Also, the same idea works for any stock path (but the actual values of (ϕ, ψ) differ usually for different paths. However, no matter which path the stock takes, the initial money paid for the option (here the 15 dollars), is *always* sufficient to set up a risk-free portfolio. Hence, the hedging strategy is self-financing. Of course, we might need to borrow money to buy part of the stock, but we will always be able to pay back our debts at the end.

(c) Derivation of the Black-Scholes formula

The Black-Scholes Model

We are now ready to transition to the continuous world. All we need to do is to formulate a stochastic process at discrete time steps δt such that we will be able to take a meaningful limit $\delta t \to 0$. We will see that this, ultimately, will be made possible via the Central Limit Theorem and the transition of the binomial distribution to the normal distribution. However, we only can establish convergence, if we have the appropriate scaling of the 'up'- and 'down'-movements of the stock with δt. It will turn out that the following scaling is appropriate: We will derive the famous Black-Scholes formula in



four steps:

1. Characterize a stock process on a tree with N steps, such that we can take later the limit $N \to \infty$.
2. Compute the risk-neutral probability q using a Taylor expansion.
3. Use the Central Limit Theorem in order to write the option price as an expectation with respect to a Gaussian probability density.
4. Rewrite the integral in terms of the cumulative normal distribution function.

Characterization of the stock process: Consider such a process with many time steps, such that $N \cdot \delta t = T$, where T is the time of maturity of the option. In this tree, with N time-ticks, each path corresponds to a possible realization of the stock process. Let's define a random variable X_N that counts the number of 'up'-jumps. Then, of course, we will have $N - X_N$ 'down'-jumps. The value at the end node of the tree for a path with X_N 'up'-jumps and $N - X_N$ 'down'-jumps is given by

$$\begin{aligned} S_T &= S_0 \exp\Big(N\delta t \cdot \mu + \sigma\sqrt{\delta t}(X_N - (N - X_N))\Big) \\ &= S_0 \exp\left(\mu T + \sigma\sqrt{T}\frac{2X_N - N}{\sqrt{N}}\right) \qquad (9) \end{aligned}$$

Note that, under the above assumptions, the value of the stock at the end node only depends on the number of 'up' and 'down' movements, not on the particular path that the stock took. Therefore the random variable X_N is sufficient to characterize all possible scenarios of the stock path.

Taylor expansion of the risk-neutral probability: As usual, we obtain the risk-neutral probability using the basic formula

$$q = \frac{s_{now} \mathrm{e}^{r\delta t} - s_d}{s_u - s_d} = \frac{\mathrm{e}^{r\delta t} - \mathrm{e}^{\mu\delta t - \sigma\sqrt{\delta t}}}{\mathrm{e}^{\mu\delta t + \sigma\sqrt{\delta t}} - \mathrm{e}^{\mu\delta t - \sigma\sqrt{\delta t}}}$$

Now we make use of the fact that δt is small. The resulting approximation for q is slightly tricky to compute as the terms above involve $\sqrt{\delta t}$, but we already did this earlier and the final result of the computation is

$$q \approx \frac{1}{2}\left(1 - \sqrt{\delta t}\left(\frac{\mu + \frac{\sigma^2}{2} - r}{\sigma}\right)\right). \tag{10}$$

Application of the Central Limit Theorem: Let us write

$$S_T = S_0\,\mathrm{e}^Y, \qquad Y = \mu T + \sigma\sqrt{T}\left(\frac{2X_N - N}{\sqrt{N}}\right) \tag{11}$$

Clearly, X_N has a binomial distribution and, therefore, under the measure $\mathbb{Q}$ we have

$$\mathbb{E}_{\mathbb{Q}}(X_N) = \mathbb{E}(X_N) = Nq, \qquad \mathrm{Var}(X_N) = Nq(1-q) \tag{12}$$

We compute now the expectation and the variance of Y. First, we see that

$$\begin{aligned}
\mathbb{E}\left(\frac{2X_N - N}{\sqrt{N}}\right) &= \frac{2Nq - N}{\sqrt{N}} \\
&= -\frac{\sqrt{\delta t}}{\sqrt{N}}\left(\frac{\mu - r + \sigma^2/2}{\sigma}\right)\cdot N \\
&= -\sqrt{T}\left(\frac{\mu - r + \sigma^2/2}{\sigma}\right)
\end{aligned}$$

Therefore, for the expectation of Y (under the measure $\mathbb{Q}$), we find

$$\begin{aligned}
\mathbb{E}(Y) &= \mu T - \sigma\sqrt{T}\sqrt{T}\left(\frac{\mu - r + \sigma^2/2}{\sigma}\right) \\
&= (r - \sigma^2/2)T
\end{aligned}$$

The variance, on the other hand, is slightly easier to compute in this approximation as we need to take into account only the leading order term ($q \approx 1/2$). Thus we find

$$\mathrm{Var}\left(\frac{2X_N - N}{\sqrt{N}}\right) \approx N \cdot \frac{1}{2}\left(1 - \frac{1}{2}\right) \cdot \frac{4}{N} = 1\,. \tag{13}$$

For the variance of Y we obtain

$$\mathrm{Var}(Y) = \sigma^2 T\,. \tag{14}$$

Therefore, as the time to maturity T is fixed, we can now take the limit $N \to \infty$ corresponding to $\delta t \to 0$. In this limit, the binomial distribution tends to a normal distribution and we can write S_T therefore as

$$S_T \approx S_0\, \mathrm{e}^{(r-\sigma^2/2)T + \sigma\sqrt{T}Z}, \qquad Z \sim N(0,1)\,. \tag{15}$$

We are now ready to write the option price as an expectation value: For a European call option $X = (S_T - K)^+$ the expectation of the discounted claim is written as

$$\begin{aligned} V &= \mathbb{E}\left(\mathrm{e}^{-rT}X\right) = \mathbb{E}\left((\mathrm{e}^{-rT}S_T - K\mathrm{e}^{-rT})^+\right) \\ &= \mathbb{E}\left((S_0\,\mathrm{e}^{\sigma\sqrt{T}Z - \sigma^2 T/2} - K\mathrm{e}^{-rT})^+\right) \end{aligned}$$

Rewriting the integral using the cumulative normal distribution: The above expectation value can be rewritten by using the cumulative normal distribution Φ defined as

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\, dy\,. \tag{16}$$

Theorem. (Black-Scholes Option Pricing) *The price V of a European call option with strike price K is given by*

$$\begin{aligned} V &= \mathbb{E}\left((S_0\,\mathrm{e}^{\sigma\sqrt{T}Z - \sigma^2 T/2} - K\mathrm{e}^{-rT})^+\right) \\ &= S_0\Phi(d_1) - K\,\mathrm{e}^{-rT}\,\Phi(d_2) \end{aligned} \tag{17}$$

with

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\,. \tag{18}$$

Proof. In order to compute the above integral, let's first write

$$V = \int_{-\infty}^{\infty} p(z) \left(S_0 \, \mathrm{e}^{\sigma\sqrt{T}z-\sigma^2 T/2} - K\mathrm{e}^{-rT}\right)^+ dz \,, \qquad p(z) = \frac{1}{\sqrt{2\pi}} \, \mathrm{e}^{-z^2/2} \,.$$

Now we note that

$$\left(S_0 \, \mathrm{e}^{\sigma\sqrt{T}z-\sigma^2 T/2} - K\mathrm{e}^{-rT}\right)^+ = \max\left(S_0 \, \mathrm{e}^{\sigma\sqrt{T}z-\sigma^2 T/2} - K\mathrm{e}^{-rT}, 0\right) \,.$$

For values of z that are negative and large in absolute value, the contribution will be zero as the first term will be negative. However, as z increases, we see, that there is a 'critical' $z = z_c$ such that for $z > z_c$, there will be non-zero contributions to the integral. This z_c is determined by the equation

$$S_0 \, \mathrm{e}^{\sigma\sqrt{T}z_c-\sigma^2 T/2} = K\mathrm{e}^{-rT} \tag{19}$$

Solving this equation for z_c yields after a couple of lines of algebra

$$z_c = \frac{\ln(K/S_0) + (\sigma^2/2 - r)T}{\sigma\sqrt{T}} \,. \tag{20}$$

This allows us to rewrite the integral for the option price V by

$$V = S_0 \int_{z_c}^{\infty} \mathrm{e}^{\sigma\sqrt{T}z-\sigma^2 T/2} p(z) \, dz - K\mathrm{e}^{-rT} \int_{z_c}^{\infty} p(z)dz \tag{21}$$

For the second integral, we find immediately ($\tilde{z} = -z$)

$$\begin{aligned} \int_{z_c}^{\infty} p(z)dz &= \int_{-\infty}^{-z_c} p(\tilde{z})d\tilde{z} = \Phi(-z_c) \\ &= \Phi\left(\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) = \Phi(d_2) \end{aligned}$$

The first integral is left as an exercise. □

Exercises

1. (5 points) Consider again a bookmaker taking bets on a horse race. Assume a race with n horses, and an amount A_k bet on the k-th horse. What odds shall the bookmaker quote for each horse? What are the implied probabilities for the horses to win?

2. (5 points) Consider a stock process S on a tree. S starts at time $i = 0$ at $S_0 = 20$ and it can go up or down 18% every year. The interest rate is 6%.

   (a) Sketch the stock process for two years.

   (b) Consider a European call option at a strike price of $K = 25$ and with an expiration date of two years. Find the value of the option at all nodes of the tree.

3. (5 points) A stock process starts at time 0 at 100 and then can go 30 up or 10 down for each time step. We consider two time steps, the risk-free interest rate is zero. Consider a European call option with strike price of $100.

   (a) Sketch the Stock process.

   (b) Find the option value at all nodes of the tree.

   (c) Assume that the stock first goes down and then goes up. Compute the necessary holdings (ϕ, ψ) of stock and bond at each time step to hedge the above option.

4. (10 points) Consider a stock process S on a tree. S starts at time $i = 0$ at $S_0 = 77$ and it can go up or down 30% every year. The interest rate is 2%.

   (a) Sketch the stock process for two years.

   (b) Consider a European call option at a strike price of $K = 88$ and with an expiration date of two years. Find the value of the option at all nodes of the tree.

   (c) Consider now a stock process following the Black-Scholes model with a volatility of 30%. If all other parameters are the same as before in the discrete model, what will be the price of the option according to the Black-Scholes formula?

5. (10 points) Use a Taylor expansion to show that indeed

$$q \approx \frac{1}{2}\left(1 - \sqrt{\delta t}\left(\frac{\mu + \frac{\sigma^2}{2} - r}{\sigma}\right)\right).$$

6. (10 points) Compute the second integral to complete the proof of the Black-Scholes formula

$$\begin{aligned} V &= \mathbb{E}_{\mathbb{Q}}\left[\left(S_0 \exp(\sigma\sqrt{T}Z - \frac{\sigma^2}{2}T) - k\exp(-rT)\right)^+\right] \\ &= S_0\Phi\left(\frac{\log(S_0/k) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - k\mathrm{e}^{-rT}\Phi\left(\frac{\log(S_0/k) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \end{aligned}$$

where Φ is the cumulative normal distribution function defined as $\Phi(x) = \mathbb{Q}(Z \leq x)$.