Problem 1 (10 points)
Consider a particle moving in the (x, y)-plane in an electromagnetic field with the Lagrangian

$$L = \frac{1}{2}m\dot{r}^2 - e(\phi - \dot{r} \cdot A)$$

Consider the case of a uniform magnetic field in z-direction, $B = (0, 0, B_z)$

1. Find the vector potential A
2. Find the Hamiltionian H and write down Hamilton's equations.
3. Show by solving the equations of motion that the particle moves in circles with the frequency ω. Find ω.

Problem 2 (10 points)
Prove Liouville's Theorem for a $2n$-dimensional Hamiltonian system. For this puropose, consider the transform

$$\tilde{q}_i = q_i + \frac{\partial H}{\partial p_i}dt, \qquad \tilde{p}_i = p_i - \frac{\partial H}{\partial q_i}dt$$

and show that for

$$d\tilde{V} = d\tilde{q}_1...\tilde{q}_n d\tilde{p}_1...\tilde{p}_n = \det(J)dV$$

the determinant of the Jacobian J is given by $\det(J) = 1 + \mathcal{O}(dt^2)$.
Hint: Show first $\det(1 + \epsilon M) = 1 + \epsilon\,\mathrm{tr}(M) + \mathcal{O}(\epsilon^2)$.

Problem 3 (10 points)
A mass m is moving under the influence of $\vec{F}(\vec{r}) = -k\vec{r}$ $(k > 0)$ on the surface of a cylinder with a fixed radius R. The cylinder's symmetry axis goes through the origin.

1. Find the Lagrange function in cylindrical coordinates. Use a Legendre transformation to find the Hamilton function.
2. Find Hamilton's equations and solve them for arbitrary initial conditions.