Problem 1 (10 points)

By applying the methods of the calculus of variations, show that if there is a Lagrangian of the form $L(q_i, \dot{q}_i, \ddot{q}_i, t)$, and Hamilton's principle holds with the zero variation of both q_i and $\dot{q}_i$ at the endpoints, then the corresponding Euler-Lagrange equations are

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial L}{\partial q_i} = 0$$

Apply this to the Lagrangian

$$L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2$$

Do you recognize the equation of motion?

Problem 2 (10 points)

Consider the motion in a central force field.

1. Starting from energy conservation given by

$$\frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r) = E$$

and conservation of the angular momentum $l = \mu r^2\dot{\phi}$, show that the following relationship of the force f and $u(\phi) = 1/r(\phi)$ holds:

$$u'' + u = -\frac{\mu}{l^2}\frac{1}{u^2}f\left(\frac{1}{u}\right))$$

2. A particle moves on a trajectory given by $r = k\phi^2$. Find the force that creates this motion and the corresponding potential $V(r)$ such that $\lim_{r\to\infty} V(r) = 0$.

3. Find the total energy of the particle on this trajectory.

Problem 3 (10 points)

The orbit of the planet mercury has an eccentricity 0.206 and a period of 0.241 years; moreover the perihelion advances slowly at a rate of 43 seconds of arc per century. One possible explanation of this effect is that the potential energy around the sun has the form

$$V(r) = \frac{-mMG}{r}\left(1 + \alpha\frac{GM}{rc^2}\right)$$

where α is a dimensionless constant and $MG/c^2 = 1.475\text{km}$ characterizes the sun's gravitational field. Demonstrate that the resulting orbit represents a precessing ellipse. Find the magnitude and sign of α needed to fit the observed data.