Problem 1 (10 points)

Consider a mathematical pendulum with a point mass m and a length l. Let θ be the angular displacement from equilibrium and θ_0 the maximum of θ.

1. Show that the period T of the oscillation is given by

$$\frac{T}{2} = \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\frac{2}{ml^2}(E - U(\theta))}}, \qquad U(\theta) = mgl(1 - \cos\theta).$$

2. Use this result to show that we can express T as

$$T = T_0 \frac{\sqrt{2}}{\pi} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}, \qquad T_0 = 2\pi\sqrt{\frac{l}{g}} \tag{1}$$

3. Using a Taylor expansion of cos up to $\mathcal{O}(\theta^4)$, show that T can be approximated by

$$T \approx T_o \left(1 + \frac{\theta_0^2}{16}\right)$$

Problem 2 (10 points)

A particle of mass m moves in a one-dimensional force field with potentials (a) $V(x) = -V_0/\cosh^2(ax)$, and (b) $V(x) = V_0 \tan^2(ax)$, $V_0 > 0$. For each case, sketch the potential and state the range of energy and position within which bounded oscillations can occur. Determine the period of oscillation as a function of the energy E.

Problem 3 (10 points)

Consider the following equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_0 \sin(\omega t)$$

1. Show that a particular solution of this equation is given by

$$x(t) = A(\omega)\sin\left(\omega t + \phi(\omega)\right).$$

   Find the *real* functions $A = A(\omega)$ and $\phi = \phi(\omega)$.

2. Sketch the graph of A^2.

3. Find the maximum of A^2.