Heavy Symmetric Top

Angular velocity:

$$\omega = (\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)\, e_1 + (\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)\, e_2 + (\dot\psi + \dot\phi \cos\theta)\, e_3$$

Lagrangian: $I_1 = I_2$

$$L = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 - Mgl\cos\theta$$

$$= \frac{1}{2} I_1 (\dot\phi^2 \sin^2\theta \sin^2\psi + \dot\theta^2 \cos^2\psi + 2\dot\phi\dot\theta \sin\theta \sin\psi\cos\psi + \dot\phi^2 \sin^2\theta\cos^2\psi + \dot\theta^2 \sin^2\psi - 2\dot\phi\dot\theta \sin\theta\sin\psi\cos\psi)$$

$$+ \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta$$

$$\Rightarrow \quad L = \frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta$$

1./ Three conserved quantities

Energy $E = T + V$, ϕ and ψ are cyclic.

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3 (\dot\psi + \dot\phi\cos\theta) = I_3 \omega_3$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \sin^2\theta\, \dot\phi + I_3 (\dot\psi + \dot\phi\cos\theta)\cos\theta$$

Heavy Symmetric Top

$$p_\psi = I_3 \dot{\psi} + I_3 \dot{\phi} \cos\theta$$

$$p_\phi = \left(I_1 \sin^2\theta + I_3 \cos^2\theta\right) \dot{\phi} + I_3 \dot{\psi} \cos\theta$$

solve for $\dot{\phi}$ and $\dot{\psi}$

$$p_\psi - I_3 \dot{\psi} = I_3 \dot{\phi} \cos\theta \qquad \dot{\phi} = \frac{p_\psi - I_3 \dot{\psi}}{I_3 \cos\theta}$$

$$\frac{p_\phi - I_3 \dot{\psi} \cos\theta}{I_1 \sin^2\theta + I_3 \cos^2\theta} = \dot{\phi} = \frac{p_\psi - I_3 \dot{\psi}}{I_3 \cos\theta}$$

$$p_\phi I_3 \cos\theta - I_3^2 \dot{\psi} \cos^2\theta = \left(p_\psi - I_3 \dot{\psi}\right)\left(I_1 \sin^2\theta + I_3 \cos^2\theta\right)$$

$$= p_\psi \left(I_1 \sin^2\theta + I_3 \cos^2\theta\right) - I_3 \dot{\psi} \left(I_1 \sin^2\theta + I_3 \cos^2\theta\right)$$

$$p_\phi I_3 \cos\theta = p_\psi \left(I_1 \sin^2\theta + I_3 \cos^2\theta\right) - I_3 \dot{\psi} I_1 \sin^2\theta$$

$$\frac{p_\phi I_3 \cos\theta - p_\psi \left(I_1 \sin^2\theta + I_3 \cos^2\theta\right)}{-I_1 I_3 \sin^2\theta} = \dot{\psi}$$

Heavy Symmetric Top

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{\left(p_\phi I_3 - p_\psi I_3 \cos\theta\right)\cos\theta}{I_1 I_3 \sin^2\theta}$$

$$= \frac{p_\psi}{I_3} - \frac{\left(p_\phi - p_\psi \cos\theta\right)\cos\theta}{I_1 \sin^2\theta}$$

Note: we can compare this to Tong's solution.

$$a = \frac{I_3 \omega_3}{I_1} = \frac{p_\psi}{I_1} \qquad b = \frac{p_\phi}{I_1}$$

$$\boxed{\dot{\psi} = \frac{I_1 a}{I_3} - \frac{\left(b - a\cos\theta\right)\cos\theta}{\sin^2\theta}}$$

$$\dot{\phi} = \frac{p_\psi}{I_3 \cos\theta} - \frac{1}{\cos\theta}\left(\frac{I_1 a}{I_3} - \frac{\left(b - a\cos\theta\right)\cos\theta}{\sin^2\theta}\right)$$

$$= \frac{I_1 a}{I_3 \cos\theta} - \frac{I_1 a}{I_3 \cos\theta} + \frac{\left(b - a\cos\theta\right)}{\sin^2\theta}$$

$$\boxed{\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}}$$

Heavy Symmetric Top

Now recall the equation for E:

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \dot{\phi}^2 \sin^2\theta + \underbrace{\frac{1}{2} I_3 \big(\overbrace{\dot{\psi} + \dot{\phi}\cos\theta}^{\omega_3}\big)^2}_{\text{const.}}$$

$$+ Mgl\cos\theta$$

(can be dropped)

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \dot{\phi}^2 \sin^2\theta + Mgl\cos\theta = \text{const.}$$

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \sin^2\theta \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)^2 + Mgl\cos\theta$$

$$= \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

$$= \frac{1}{2} I_1 \dot{\theta}^2 + V_{eff}(\theta)$$

with $$V_{eff}(\theta) = \frac{I_1 (b - a\cos\theta)^2}{2\sin^2\theta} + Mgl\cos\theta$$

$$u = \cos\theta$$

$$\dot{u} = -\dot{\theta}\sin\theta \qquad \dot{\theta} = \frac{-\dot{u}}{\sin\theta} \qquad \dot{\phi}^2 = \frac{\dot{u}^2}{\sin^2\theta} = \frac{\dot{u}^2}{1-u^2}$$

$$E' = \frac{1}{2} I_1 \frac{\dot{u}^2}{1-u^2} + \frac{I_1 (b - au)^2}{2(1-u^2)} + Mglu$$

Heavy Symmetric Top

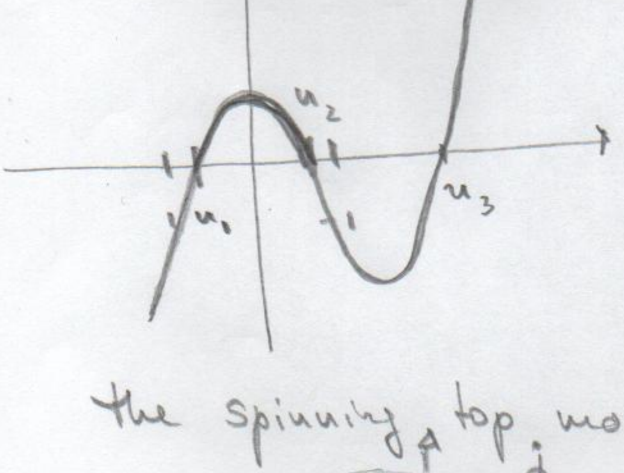
$$\frac{2E'(1-u^2)}{I_1} - (b-au)^2 - \frac{Mglu\cdot 2(1-u^2)}{I_1} = \dot{u}^2$$

$$(1-u^2)\left(\frac{2E'}{I_1} - \frac{2Mgl}{I_1}u\right) - (b-au)^2 = \dot{u}^2$$

$$\boxed{\dot{u}^2 = (1-u^2)(\alpha-\beta u) - (b-au)^2}$$

with $\alpha = \frac{2E'}{I_1}$ and $\beta = \frac{2Mgl}{I_1}$ and $-1 \leq u \leq 1$

$0 \leq \dot{u}^2 = f(u)$ where $f(u)$ is the cubic polynomial above.



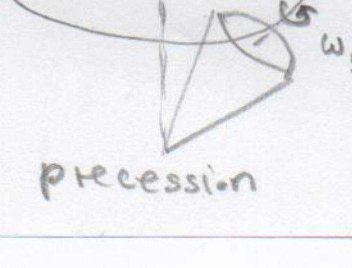
$f(\pm 1) = -(b \mp a)^2 < 0$

motion between u_1 and u_2

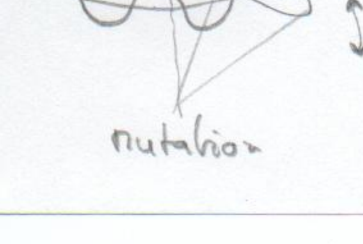
($\dot{u}^2 = f(u)$ needs to be ≥ 0)

If $\dot{\phi} > 0$ at u_1 and u_2

the spinning top moves "forward" in $\dot{\phi}$



needs "wobble"



Heavy Symmetric Top

Note that $\dot{\phi} = \dfrac{b - a u}{1 - u^2}$

If $\dot{\phi} > 0$ at $u = u_1$ and $\dot{\phi} = 0$ at $u = u_2$

we obtain

We can create this motion if we let the top go at an angle.

$$f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2$$

Stability of "Sleeping Top"

spin upright, $\theta = 0$ and $\dot{\theta} = 0$. $\dot{u}^2 = f(u)$

Since $\dot{\theta} = 0$ implies $\dot{u} = 0$, $f(u)$ needs to be zero at this point. $\theta = 0 \Rightarrow u = \cos\theta = 1$, hence $f(1) = 0$

$$f(1) = -(b-a)^2 = 0 \Rightarrow a = b$$

Also, we had $\alpha = \dfrac{2E'}{I_1}$ and $\beta = \dfrac{2Mgl}{I_1}$

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \dot{\phi}^2 \sin^2\theta + Mgl \cos\theta$$

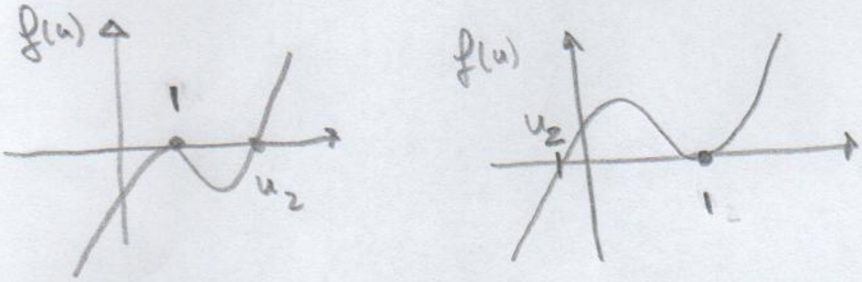
at $\theta = 0$, $\dot{\theta} = 0$ we have $E' = Mgl$, hence $\alpha = \beta$

Heavy Symmetric Top

$$f(u) = \alpha(1-u^2)(1-u) - a^2(1-u)^2$$

$$= (1-u)^2\left(\alpha(1+u) - a^2\right)$$ so f has a double zero at $u_1 = 1$ and $u_2 = \dfrac{a^2}{\alpha} - 1$

Now u_2 can be large or smaller u_1



$$a = \frac{I_3 \omega_3}{I_1}$$

$$\alpha = \frac{2E'}{I_1} = \frac{2Mgl}{I_1} = \beta$$

If $u_2 > 1$, then stable, if $u_2 < 1$ unstable.

$$\frac{a^2}{\alpha} - 1 > 1 \quad \text{or} \quad \frac{a^2}{\alpha} > 2 \qquad \frac{\dfrac{I_3^2 \omega_3^2}{I_1^2}}{\dfrac{2Mgl}{I_1}} > 2$$

or $$\omega_3^2 > \frac{4MglI_1}{I_3^2}$$ for stability.