Invariance of Maxwell's wave equation:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

coordinate transform: $x = \gamma\,(x' + v t')$
$t = \gamma\,(t' + \frac{v}{c^2} x')$

$$f'(x',t') = f(x(x',t'),\ t(x',t'))$$

$$\Rightarrow \frac{\partial f'}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x'} = \frac{\partial f}{\partial x}\gamma + \frac{\partial f}{\partial t}\gamma\frac{v}{c^2}$$

$$\frac{\partial f'}{\partial t'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial t'} = \frac{\partial f}{\partial x} v\gamma + \frac{\partial f}{\partial t}\gamma$$

$$\Rightarrow \frac{\partial^2 f'}{\partial x'^2} = \frac{\partial^2 f}{\partial x^2}\gamma^2 + 2\frac{\partial^2 f}{\partial x \partial t}\gamma^2\frac{v}{c^2} + \frac{\partial^2 f}{\partial t^2}\gamma^2\frac{v^2}{c^4}$$

$$\frac{\partial^2 f'}{\partial t'^2} = \frac{\partial^2 f}{\partial x^2} v^2\gamma^2 + 2\frac{\partial^2 f}{\partial x\partial t} v\gamma^2 + \frac{\partial^2 f}{\partial t^2}\gamma^2$$

$$\Rightarrow \frac{\partial^2 f'}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 f'}{\partial t'^2} = \frac{\partial^2 f}{\partial x^2}\left(\gamma^2 - \gamma^2\frac{v^2}{c^2}\right) - \frac{1}{c^2}\frac{\partial^2 f'}{\partial t^2}\left(\gamma^2 - \gamma^2\frac{v^2}{c^2}\right)$$

$$= 0$$

$\square$

Example of a collision problem :

$$p + p \longrightarrow p + p + p + \bar{p}$$

fast proton hits proton at rest → proton - antiproton creation

We need (at least) $4 m_0 c^2$ on r.h.s.

Remember: $E^2 - p^2c^2 = m^2c^4$

$$(mc^2 + m_0c^2)^2 - \underset{\substack{\uparrow \\ \text{minimum momentum}}}{p^2c^2} = (4m_0c^2)^2$$

Incident proton: $m^2c^4 - p^2c^2 = (m_0c^2)^2$

$$\begin{aligned} m^2c^4 + 2mm_0c^4 + m_0^2c^4 - p^2c^2 &= 16\, m_0^2c^4 \\ m^2c^4 \qquad\qquad\qquad\qquad\qquad - p^2c^2 &= m_0^2c^4 \end{aligned}$$

$$\Rightarrow \quad 2mm_0c^4 = 14\, m_0^2c^4 \Rightarrow m = 7m_0$$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 7 \Rightarrow \boxed{v = \sqrt{\frac{48}{49}}\; c}$$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

Given $L = -m_0 c^2 \gamma^{-1} - q\phi + \frac{q}{c}\vec{v}\cdot\vec{A}$, find the Hamiltonian H.

<u>Solution:</u> $H = \sum_i \dot{x}_i p_i - L$, $p_i = \frac{\partial L}{\partial \dot{x}_i}$, $\vec{v} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$

$$\left(\frac{1}{\gamma}\right)^2 = \frac{1}{\gamma^2} = 1 - \frac{\dot{x}_1^2 + \dot{x}_1^2 + \dot{x}_3^2}{c^2} \qquad \frac{\partial}{\partial \dot{x}_i}\left(\frac{1}{\gamma}\right) = ?$$

$$2\cdot\frac{1}{\gamma}\frac{\partial}{\partial \dot{x}_i}\left(\frac{1}{\gamma}\right) = -\frac{\dot{x}_i}{c^2}\cdot 2 \Rightarrow \frac{\partial}{\partial \dot{x}_i}\left(\frac{1}{\gamma}\right) = -\frac{\gamma \dot{x}_i}{c^2}$$

$$\vec{v}\cdot\vec{A} = \sum_i \dot{x}_i A_i \Rightarrow p_i = \frac{\partial L}{\partial \dot{x}_i} = m_0 \gamma \dot{x}_i + \frac{q A_i}{c}$$

$$\Rightarrow H = \sum_i \dot{x}_i p_i - L = m_0 \gamma \sum_i \dot{x}_i^2 + \frac{m_0 c^2}{\gamma} + q\phi$$

$$= \frac{m_0 c^2}{\gamma}\left(\frac{\gamma^2 v^2}{c^2} + 1\right) + q\phi = m_0 c^2 \gamma \underbrace{\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right)}_{=1} + q\phi$$

$$\Rightarrow \quad H = m_0 \gamma c^2 + q\phi$$

Now in terms of p:

$$\sum_i (m_0 \gamma \dot{x}_i)^2 = \sum_i \left(p_i - \frac{qA_i}{c}\right)^2$$

$$m_0^2 \gamma^2 v^2 = \left(\vec{p} - \frac{q\vec{A}}{c}\right)^2$$

$$m_0^2 \gamma^2 c^4 = m_0^2 \left(\frac{\gamma^2 v^2}{c^2} + 1\right) c^4 = \left(\vec{p} - \frac{q\vec{A}}{c}\right)^2 c^2 + m_0^2 c^4$$

$$H = \sqrt{\left(\vec{p} - \frac{q\vec{A}}{c}\right)^2 c^2 + m_0^2 c^4} + q\phi$$

Euler-Lagrange with curvature:

$$L = \frac{1}{2} g_{ab}(q^1, \dots, q^n)\, \dot{q}^a \dot{q}^b$$

$g_{ab} = g_{ba}$, g^{ab} exists (inverse)

$$g^{ab} g_{bc} = \delta^a_c$$

$\Rightarrow$ Euler-Lagrange: $\ddot{q}^d + \Gamma^d_{bc}\, \dot{q}^b \dot{q}^c = 0$

$$\Gamma^d_{bc} = \frac{1}{2} g^{da} \left(\frac{\partial g_{ab}}{\partial q^c} + \frac{\partial g_{ac}}{\partial q^b} - \frac{\partial g_{cb}}{\partial q^a} \right)$$

Proof:

$$\frac{\partial L}{\partial \dot{q}^a} = g_{ab} \dot{q}^b \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} = g_{ab}\ddot{q}^b + \dot{q}^b \frac{\partial g_{ab}}{\partial q^c} \dot{q}^c$$

$$= g_{ab}\ddot{q}^b + \frac{1}{2}\left(\frac{\partial g_{ab}}{\partial q^c} \dot{q}^b \dot{q}^c + \frac{\partial g_{ac}}{\partial q^b} \dot{q}^c \dot{q}^b \right)$$

$$\Rightarrow g_{ab}\ddot{q}^b + \frac{1}{2}\left(\frac{\partial g_{ab}}{\partial q^c} \dot{q}^c \dot{q}^b + \frac{\partial g_{ac}}{\partial q^b} \dot{q}^b \dot{q}^c - \frac{\partial g_{bc}}{\partial q^a} \dot{q}^b \dot{q}^c \right) = 0$$

$$g^{da} g_{ab} = \delta^d_b .$$

$$\Rightarrow \ddot{q}^d + \frac{1}{2} g^{da} \left(\frac{\partial g_{ab}}{\partial q_c} \dot{q}^c \dot{q}^b + \frac{\partial g_{ac}}{\partial q_b} \dot{q}^b \dot{q}^c - \frac{\partial g_{bc}}{\partial q^a} \dot{q}^b \dot{q}^c \right) = 0$$

$$\ddot{q}^d + \Gamma^d_{bc} \dot{q}^b \dot{q}^c = 0$$

$$\Gamma^d_{bc} = \frac{1}{2} g^{da} \left(\frac{\partial g_{ab}}{\partial q^c} + \frac{\partial g_{ac}}{\partial q^b} - \frac{\partial g_{cb}}{\partial q^a} \right)$$

(Christoffel symbols)

Schrödinger → Hamilton-Jacobi

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial q^2} + V(q)\,\psi$$

$$\psi(q,t) = R(q,t)\, e^{iW(q,t)/\hbar} \qquad R, W \text{ are real}$$

$\Rightarrow$ $R^2 = |\psi|^2$ is the probability density.

$$i\hbar\left[R_t + \underbrace{\frac{i}{\hbar} R W_t}\right] = -\frac{\hbar^2}{2m}\left[R_{qq} + \frac{2i}{\hbar} R_q W_q - \underbrace{\frac{R}{\hbar^2}(W_q)^2} + \frac{iR}{\hbar} W_{qq}\right] + \underbrace{V(q)}\cdot R$$

limit $\hbar \to 0$: $\quad \hbar \left|\frac{\partial^2 W}{\partial q^2}\right| \ll \left|\frac{\partial W}{\partial q}\right|$ etc.

$$\Rightarrow \quad -RW_t = \frac{1}{2m}(W_q)^2 R + VR \qquad \text{or}$$

$$\boxed{\frac{1}{2m}(W_q)^2 + V(q) + W_t = 0}$$