Relativistic Dynamics

We want to formulate the laws of mechanics in invariant form:

How can we define a 4-vector describing the velocity?

$$\text{velocity} = \frac{\text{distance}}{\text{time}} \qquad d\tau^2 = \frac{ds^2}{c^2} = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$$

$$d\tau = dt\sqrt{1 - \frac{v^2}{c^2}}$$ is a Lorentz-scalar (proper time)

4-velocity: $$\boxed{u^\mu = \frac{dx^\mu}{d\tau}}$$

Check: For $\beta = \frac{v}{c} \to 0$ we recover the Newtonian limit:

$$u^1 = \frac{dx^1}{d\tau} = \frac{dx}{\sqrt{1-\frac{v^2}{c^2}}\,dt} \longrightarrow v_x = \frac{dx}{dt} \quad \text{for } \beta \to 0$$

Note: $u^0 = \frac{dx^0}{d\tau} = \frac{c\,dt}{\sqrt{1-\beta^2}\,dt} = \frac{c}{\sqrt{1-\beta^2}}$

4-acceleration: $a^\mu = \frac{du^\mu}{d\tau}$

How can we rewrite Newton's law?

we would like $m_0 \frac{du^\mu}{d\tau} = F^\mu$ and we need to find the appropriate F^μ. Spatial components: $m_0 \frac{du^i}{d\tau} = F^i$

$$\frac{m_0}{\sqrt{1-\beta^2}} \frac{du^i}{dt} = \frac{m_0}{\sqrt{1-\beta^2}} \frac{d}{dt}\left(\frac{dx^i/dt}{\sqrt{1-\beta^2}}\right) = F^i$$

$$m_0 \frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}} \frac{dx^i}{dt}\right) = F^i \sqrt{1-\beta^2} \equiv F^i_{\text{Newton}}$$

$$\Rightarrow (F^1, F^2, F^3) = \frac{1}{\sqrt{1-\beta^2}} \vec{F}_{\text{Newton}}$$

What about the zero-component?

$$m_0 \frac{du^0}{d\tau} = F^0$$

Note: $$\frac{d}{dt}(u_\nu u^\nu) = \frac{d}{dt}(\eta_{\mu\nu} u^\mu u^\nu) = 0$$

as $\eta_{\mu\nu} u^\mu u^\nu = c^2$ (true in rest frame)

$$\Rightarrow \quad \eta_{\mu\nu} \frac{du^\mu}{d\tau} u^\nu = 0 \Rightarrow \eta_{\mu\nu} F^\mu u^\nu = 0 \quad \text{or}$$

$$u^0 F^0 - u^1 F^1 - u^2 F^2 - u^3 F^3 = 0$$

$$\frac{c}{\sqrt{1-\beta^2}} F^0 - \frac{\vec{F}_{Newton} \cdot \vec{v}}{(1-\beta^2)} = 0$$

$$\Rightarrow \quad F^0 = \frac{1}{c} \frac{\vec{F}_{Newton} \cdot \vec{v}}{\sqrt{1-\beta^2}}$$. Now recall $m_0 \frac{du^0}{d\tau} = F^0$

$$\Rightarrow \quad \frac{m_0}{\sqrt{1-\beta^2}} \frac{d}{dt}\left(\frac{c}{\sqrt{1-\beta^2}}\right) = \frac{1}{c} \frac{\vec{F}_{Newton} \cdot \vec{v}}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{m_0 c^2}{\sqrt{1-\beta^2}}\right) = \vec{F}_{\text{Newton}} \cdot \vec{v}$$

$\uparrow$

change of energy

$$\frac{d}{dt}(E) = \vec{F}_{\text{Newton}} \cdot \vec{v} \quad \Rightarrow$$

$$\boxed{E = \frac{m_0 c^2}{\sqrt{1-\beta^2}} = mc^2} \qquad m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Small v: $E = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

$$\approx m_0 c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = m_0 c^2 + \frac{1}{2} m_0 v^2$$

Relativistic momentum: $\vec{p} = m_0 \gamma \vec{v} = (p^1, p^2, p^3)$

$$p^\mu = m_0 u^\mu \quad p^0 = E/c \quad \left(\frac{m_0 c}{\sqrt{1-\beta}} \text{ for zero-component}\right)$$

$$p_\mu p^\mu = \eta_{\nu\mu} p^\nu p^\mu = \frac{E^2}{c^2} - \vec{p}^{\,2} = \frac{m_0^2 c^4}{c^2}$$

evaluated in rest frame: $p^0 = m_0 c$

$$\Rightarrow \quad \boxed{E^2 = m_o^2 c^4 + \vec{p}^{\,2} c^2}$$

Lagrange function :

$L(\vec{r}, \dot{\vec{r}}, t)$ such that this will reproduce

$$\vec{F}_{Newton} = \frac{d}{dt}(m\gamma \vec{v}) = -\nabla U$$

for a potential U.

$$L = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} - U$$

Why? $\frac{\partial L}{\partial r^k} = -\frac{\partial U}{\partial r^k}$, $\vec{r} = (r^1, r^2, r^3) \equiv (x, y, z)$

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

$$\frac{\partial L}{\partial \dot{r}_k} = \frac{dL}{dv} \frac{\partial v}{\partial \dot{r}^k} =$$

$$\frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}} \cdot \frac{\dot{r}^k}{v} = \frac{m_o \dot{r}^k}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{m_0 \dot{r}^k}{\sqrt{1-\frac{v^2}{c^2}}}\right) = -\frac{\partial U}{\partial r} \quad \text{as desired.}$$

<u>Hamiltonian:</u> $H = \vec{p}\cdot\vec{v} - L$

$$= \gamma\,(m_0\dot{x}^2 + m_0\dot{y}^2 + m_0\dot{z}^2) + m_0 c^2\sqrt{1-\frac{v^2}{c^2}} + U$$

$$\Rightarrow H = \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_0 c^2\sqrt{1-\frac{v^2}{c^2}} + U$$

$$= \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} + U = c\sqrt{m_0^2 c^2 + \vec{p}^{\,2}} + U$$

(Last step: $\sqrt{m_0^2 c^2 + m_0^2\gamma^2 v^2}$

$$= m_0 c\sqrt{1+\gamma^2\frac{v^2}{c^2}} = m_0 c\sqrt{1+\frac{v^2}{c^2}\frac{1}{1-\frac{v^2}{c^2}}}$$

$$= m_0 c\sqrt{1+\frac{v^2}{c^2-v^2}} = m_0 c\sqrt{\frac{c^2}{c^2-v^2}} = \frac{m_0 c}{\sqrt{1-\frac{v^2}{c^2}}}\ )$$

Covariant formulation:

$$S = \int_{\tau_1}^{\tau_2} \mathcal{L}(x^\mu, u^\mu, \tau)\, d\tau$$

no forces: $\frac{d}{d\tau}(m_0 u^\mu) = 0$

$$\mathcal{L} = -m_0 c \sqrt{u_\nu u^\nu} = -m_0 c \sqrt{\eta_{\mu\nu} u^\mu u^\nu}$$

will yield this result:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \quad \frac{\partial \mathcal{L}}{\partial u^\mu} = \frac{-m_0 c\, \eta_{\mu\nu} u^\nu}{\sqrt{\eta_{\nu\mu} u^\mu u^\nu}} = -m_0 u_\mu$$

$$\Rightarrow \frac{d}{d\tau}(m_0 u_\mu) = 0 \Rightarrow \frac{d}{d\tau}(m_0 u^\mu) = 0$$

This action gives the correct Newtonian limit.

$$\mathcal{L}\, d\tau = -m_0 c \sqrt{\eta_{\mu\nu} u^\mu u^\nu}\, \frac{d\tau}{dt}\, dt$$

$$= -m_0 c \sqrt{1 - \frac{v^2}{c^2}}\, dt \approx -m_0 c^2 \left(1 - \frac{v^2}{2c^2}\right) dt$$

$$= \left(-m_0 c^2 + \frac{m_0}{2} v^2\right) dt.$$