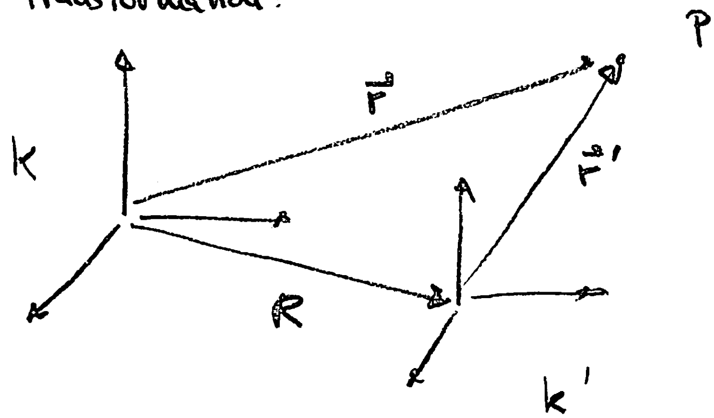
Theory of Special Relativity:

Galilei transformation:



$$\vec{r} = \vec{R} + \vec{r}\,'$$

$$\Rightarrow \quad \vec{r}\,' = \vec{r} - \vec{R}$$

$$\frac{d\vec{R}}{dt} = \vec{u}$$

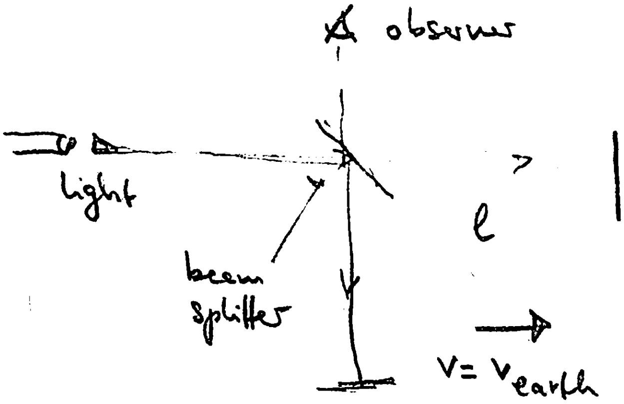
$$\Rightarrow \quad \vec{r} = \vec{r}\,' + \vec{u}\cdot t' \qquad \vec{r}\,' = \vec{r} - \vec{u}\cdot t$$

$$t = t' \qquad t' = t$$

Michelson-Morley experiment:

Assume: ether (medium in which light propagates as a wave)

$\Rightarrow$ speed (measured) should depend if measured perpendicular or parallel to rotation of the earth.

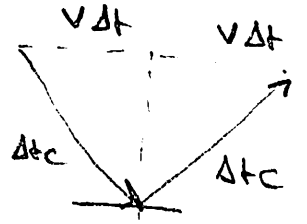


parallel: $\Delta t = \dfrac{\ell}{c-v} + \dfrac{\ell}{c+v}$

$$= \frac{2\ell c}{c^2 - v^2} = \frac{2\ell}{c}\left(\frac{1}{1-\frac{v^2}{c^2}}\right)$$

$$\approx \frac{2\ell}{c}\left(1 + \frac{v^2}{c^2}\right)$$

perpendicular:



$$\ell^2 = (\Delta t)^2 (c^2 - v^2)$$

$\Rightarrow$ light needs $\Delta t = \dfrac{\ell}{\sqrt{c^2 - v^2}}$ to go and to return

$$\Rightarrow \quad t_\perp = \frac{2\ell}{\sqrt{c^2-v^2}} = \frac{2\ell}{c}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) \approx \frac{2\ell}{c}\left(1 + \frac{v^2}{2c^2}\right)$$

$$\Rightarrow \quad t_\parallel - t_\perp \approx \frac{2\ell}{c}\,\frac{v^2}{2c^2} = \frac{\ell v^2}{c^3}$$ small, but observable

1881 1st experiment: No $\overset{\circ}{\circ}$, 1887 better exp: No $\overset{\circ\circ}{\circ\circ}$

Principles of special relativity

(1) Laws of physics are identical in all inertial frames of reference

(2) The speed of light has always the value c and is independent of the observer

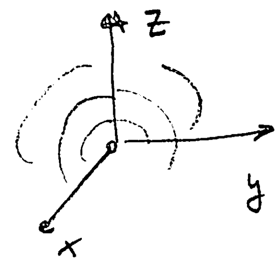
$\Rightarrow$ we need a different transformation and we have to "let go" $t' = t$.

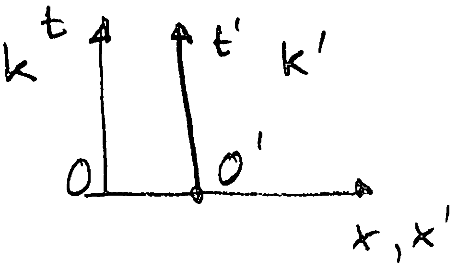
Consider a wave on a sphere:

$$x^2+y^2+z^2-(ct)^2 = 0 = x'^2+y'^2+z'^2-(ct')^2$$

Set $y=y'$, $z=z'$

for simplicity





$$x^2-(ct)^2 = x' - (ct')^2$$

Origin O' of k' observed in k satisfies

$x = vt \Rightarrow x' = \gamma(x-vt)$ (as x'-coordinate of O' is zero)

Origin O of k observed in k' satisfies

$x' = -vt' \Rightarrow x = \gamma'(x'+vt')$

$$\Rightarrow x = \gamma'(\gamma(x-vt)+vt') \quad \text{solve for } t'$$

$$\left(\frac{x}{\gamma'} - \gamma(x-vt)\right) = vt' \Rightarrow t' = \frac{1}{v}\left(\frac{x}{\gamma'} - \gamma(x-vt)\right)$$

$$\Rightarrow x^2-(ct)^2 = \gamma^2(x-vt)^2 - \left[\frac{c}{v}\left(\frac{x}{\gamma'} - \gamma(x-vt)\right)\right]^2$$

compare coeff. of x^2, xt, t^2 $\Rightarrow$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \gamma' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

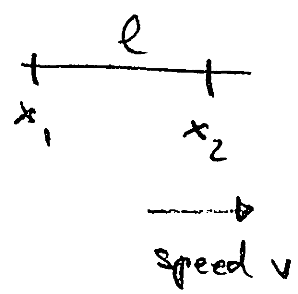
$\Rightarrow$ Lorentz - Transformation

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}} \qquad x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} \qquad t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1-\frac{v^2}{c^2}}}$$

Important consequences: (a) length contraction
(b) time dilation

(a)



in K observed in K'

$$x_2' - x_1' = \frac{x_2 - x_1 - v(t_2 - t_1)}{\sqrt{1-\frac{v^2}{c^2}}}$$

measurement in K' requires $t_2' = t_1'$

$$\Rightarrow \quad t_2 - \frac{v}{c^2}x_2 = t_1 - \frac{v}{c^2}x_1 \quad \text{or} \quad t_2 - t_1 = \frac{v}{c^2}(x_2 - x_1)$$

$$\Rightarrow \quad x_2' - x_1' = (x_2 - x_1)\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(1-\frac{v^2}{c^2}\right) = (x_2 - x_1)\sqrt{1-\frac{v^2}{c^2}}$$

$$\Rightarrow \quad |x_2' - x_1'| < |x_2 - x_1|$$

The ruler appears shorter

(b) Time dilation: Clock at x (fixed) sending signals separated by $\Delta t = t_2 - t_1$)

$$\Delta t' = t_2' - t_1' =$$

$$\frac{t_2 - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1 - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}} > \Delta t$$

(a moving clock ticks more slowly)

Transformation of velocities:

Assume that a particle moves in reference frame K in the time dt the distance dx, hence $u_x = \frac{dx}{dt}$

Assume that K' moves only in x-direction, with speed v

In K: $(x,t) \longrightarrow (x+dx,\ t+dt)$

in K': $(x',t') \longrightarrow (x'+dx',\ t'+dt')$

$$x' = \gamma(x - vt) \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$x' + dx' = \gamma\left[x + dx - v(t+dt)\right]$$

$$\Rightarrow \quad dx' = \gamma(dx - v\,dt) \qquad dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

$$u_{x'} = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

Note: $u_{y'}$ will be affected through dt' although $dy = dy'$:

$$u_{y'} = \frac{dy'}{dt'} = \frac{dy}{dt - \frac{v}{c^2}dx}\sqrt{1-\frac{v^2}{c^2}} = \frac{u_y\sqrt{1-\frac{v^2}{c^2}}}{1 - \frac{v u_x}{c^2}}$$

$\Rightarrow$ motions are not independent anymore.

How can we do the calculations efficiently?
How can we see whether a law is written in invariant form?

$\Rightarrow$ use 4-vectors

$x^0 = ct \quad x^1 = x \quad x^2 = y \quad x^3 = z$

Greek indices: 0, 1, 2, 3 Latin indices 1, 2, 3

Lorentz transformation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad , \quad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad , \quad \beta = \frac{v}{c}$$

How can we define the "length" of a 4-vector $a = (x^\mu)$? The length should be Lorentz-invariant.

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Proof:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$ds'^2 = c^2\gamma^2\left(dt - \frac{v}{c^2}dx\right)^2 - \gamma^2(dx - v\,dt)^2 - dy^2 - dz^2$$

$$= (c^2\gamma^2 - \gamma^2 v^2)\,dt^2 + \left(\gamma^2\frac{v^2}{c^2} - \gamma^2\right)dx^2 - dy^2 - dz^2$$

$$= ds^2$$

as, obviously, $c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) = c^2$ and

$$\gamma^2\left(\frac{v^2}{c^2} - 1\right) = 1$$

It is convenient to write

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu \equiv \eta_{\mu\nu}\,dx^\mu dx^\nu$$

$g_{\mu\nu} \equiv \eta_{\mu\nu}$ is called metric tensor (Minkowski metric)

A 4-vector is called <u>contravariant</u> if

$$(a^\mu) = (a^0, a^1, a^2, a^3) \quad , \quad \boxed{a'^\mu = \Lambda^\mu{}_\nu\, a^\nu}$$

What about the inverse transformation?

$$x^\mu = \Lambda_\nu{}^\mu x'^\nu \qquad \Lambda_\nu{}^\mu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_\nu{}^\mu = \frac{\partial x^\mu}{\partial x'^\nu}, \quad \Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$$

Are there any quantities that transform using $\Lambda_\nu{}^\mu$? Consider a scalar $\varphi(x^\mu)$

$$\varphi(x^\mu) = \varphi(x^\mu(x'^\nu)) = \varphi'(x'^\nu)$$

$$\Rightarrow \frac{\partial \varphi'}{\partial x'^\nu} = \frac{\partial \varphi}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\nu} = \Lambda_\nu{}^\mu \frac{\partial \varphi}{\partial x^\mu}$$

A 4-vector is called <u>covariant</u> if

$$a = (a_0, a_1, a_2, a_3)$$

$$\boxed{a_\nu' = \Lambda_\nu{}^\mu a_\mu}$$

And we can write the length as $a_\nu a^\nu = g_{\mu\nu} a^\mu a^\nu$.