Analytical Dynamics

Lecture 11

Periodic motion in higher dimensions:

$n=1$: integrability.

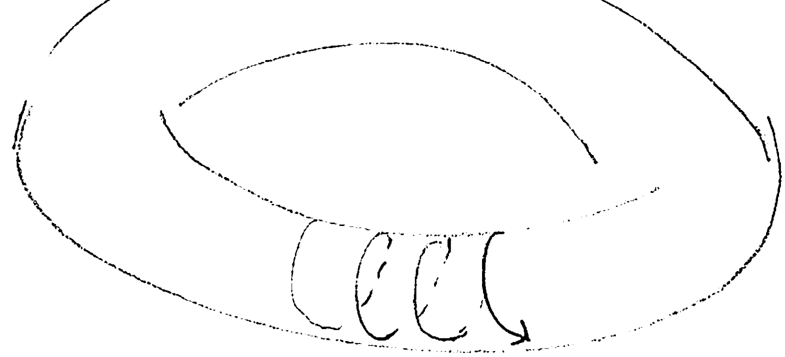
$n=2$: geometry is more complicated

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2} m_1 \omega_1^2 q_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2} m_2 \omega_2^2 q_2^2 = E$$

can be transformed to action-angle variables

$I_i = \oint p_i \, dq_i$ over complete periods in the (q_i, p_i) plane.

We have (q_1, p_1) oscillates with ω_1,
(q_2, p_2) oscillates with ω_2. If $\omega_2 > \omega_1$,



motion on a torus

if $\frac{\omega_2}{\omega_1} = \frac{p}{q} \in \mathbb{Q} \rightarrow$ closed orbit

If frequencies are incommensurable →
trajectory will slowly fill out orbit
(dense periodic orbit)

KAM - theorem

(Kolmogorov - Arnold - Moser)

If $H = H_0 + \epsilon H_1$, and H_1 is non-integrable
and distrubs a bounded motion and the
frequencies of H_0 are incommensurable

⇒ the motion remains confined to an
N-torus for most initial conditions
new trajectories close to old trajectories
(and there is systematic method to calculate
the influence of H_1: canonical perturbation
theory)

If KAM does not hold: "chaos" is possible.

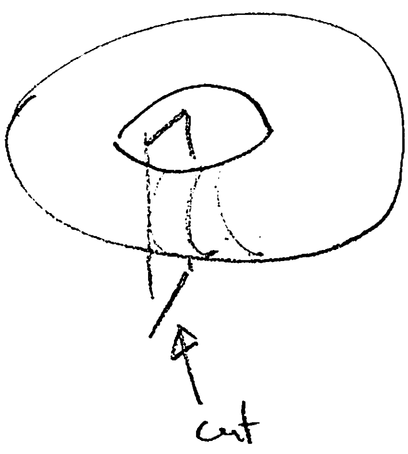
"Chaos" :

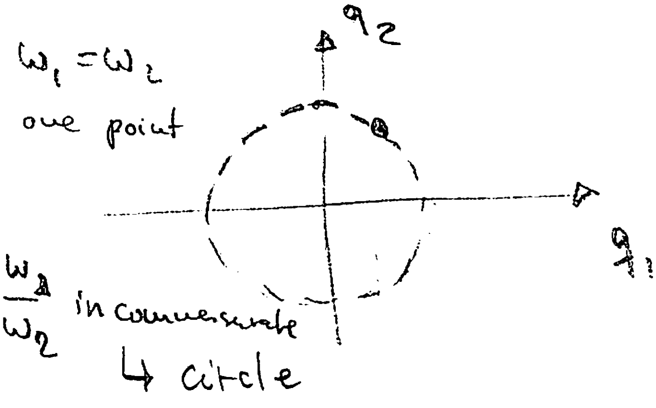
here: deterministic.

- sensitivity to initial conditions (makes it difficult to predict long-time behavior)
- often a-periodic "chaotic" motion and possibly "switching"
- often dependence on a control parameter

Poincaré map :

Easier to visualise : two-dimensional slice through the energy hypersurface given by $H(q_i, p_i) = E$





General features can be understood by the driven-damped pendulum:

$$\ddot{\theta} + \left(\frac{1}{q}\right)\dot{\theta} + \sin\theta = g \cos(\omega_D t)$$

damping — strength of forcing — frequency of forcing

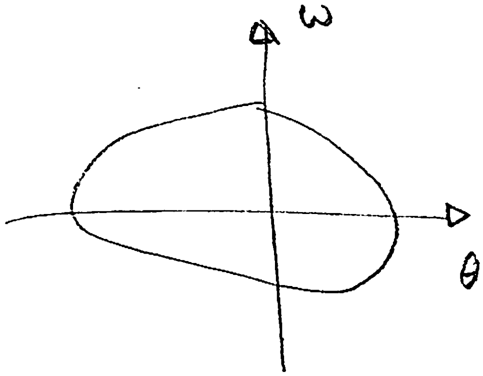
$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\frac{1}{q}\omega - \sin\theta + g\cos(\omega_D t)$$

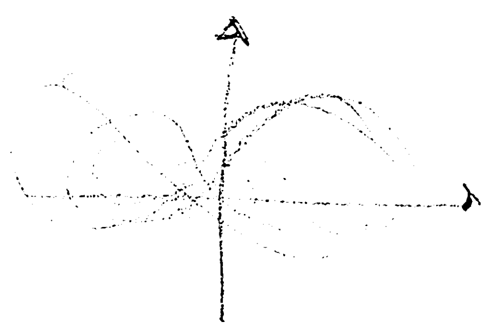
can be solved numerically:

$q = 2$, $g = 0.9$ → "normal behavior"

$q = 2$, $g = 1.15$ → "chaotic behavior"

initial condition: $\theta(0) = 2.5$, $\omega(0) = 0$





Analytical Dynamics

One way of analyzing the system is to think of "snap-shots" at period $T = \frac{2\pi}{\omega_D}$ (corresponding to Poincaré sections).

We can then think of $\begin{pmatrix} \theta_{n+1} \\ \omega_{n+1} \end{pmatrix} = G \begin{pmatrix} \theta_n \\ \omega_n \end{pmatrix}$

where the exact form of G is very complicated.

Simple example of a map: logistic map

$x_{n+1} = a x_n (1 - x_n)$ where $a > 0$ is the control parameter

$(0 \leq x \leq 1)$

Fixpoint? $x_n \to x_\infty$? Set $x_{n+1} = x_n$, hence

$x_n = a x_n - a x_n^2$ or $0 = (a-1) x_n - a x_n^2$

$\Rightarrow x_n ((a-1) - a x_n) = 0$ or $x_\infty = \frac{a-1}{a}$

Stability of this fixed point:

Set $x_n = x_\infty + \delta$

$$\Rightarrow \quad x_{n+1} = a x_n (1-x_n) = a(x_\infty+\delta)(1-x_\infty-\delta)$$
$$= a x_\infty (1-x_\infty) + a\delta(1-x_\infty) - a x_\infty \delta + \mathcal{O}(\delta^2)$$
$$= x_\infty + (a - 2a x_\infty)\,\delta$$

$$a - 2a x_\infty = a - 2(a-1) = 2-a.$$

For stability of fixed point: $|2-a| < 1$, hence $1 < a < 3$.

$\Rightarrow$ attractor. What happens for $a > 3$?

Numerics: very easy: after some iterations

$a = 3.2 \longrightarrow$ alternates between two values

$a = 3.5 \longrightarrow$ alternates between four values

$\hookrightarrow$ number of values of x_∞ doubles, until $a_\infty = 3.5699$ (Feigenbaum point) is reached.

Note: If we know that the value jumps between two values:

$$x_{n+1} = a x_n (1 - x_n)$$

$$x_n = x_{n+2} = a x_{n+1} (1 - x_{n+1})$$

$$\Rightarrow \quad x_n = a^2 x_n (1 - x_n)(1 - a x_n (1 - x_n)) \quad \text{or}$$

$$0 = x_n \left[a^2 (1 - x_n)(1 - a x_n - a x_n^2) - 1 \right] = 0$$

hence we get a cubic eq:

$$a^2 (1 - a x_n - a x_n^2 - x_n + a x_n^2 + a x_n^3) - 1 = 0 \quad \text{or}$$

$$\boxed{a^3 x_n^3 - 2 a^3 x_n^2 - a^2 (a+1) x_n + a^2 - 1 = 0}$$

Interesting systems

- Henon-Heiles Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}k(x^2+y^2) + \lambda\left(x^2y - \frac{1}{3}y^3\right)$$

$\hookrightarrow$ Goldstein

- Van der Pol's equation

$$\ddot{x} + a(x^2-1)\dot{x} + x = \beta \cos\omega t$$

- Duffing's equation

$$\ddot{x} + \delta\dot{x} - x + x^3 = \gamma \cos\omega t$$

- Lorenz equations

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = -\beta z + xy$$

One way to measure "how" chaotic the system is: Liapunov exponent.

If two orbits are separated by s_0 initially, then their separation at a later time t is

$$s(t) = s_0 e^{\lambda t}$$

$$s(n) = s_0 e^{\lambda n}$$

for maps.

$\lambda < 0 \longrightarrow$ attractor

chaos $\Rightarrow$ $\lambda > 0$

Numerics

Euler:

$$y_{n+1} = y_n + f(t_n, y_n)\,\Delta t$$

Midpoint:

$$k = \Delta t\, f(t_n, y_n)$$

$$y_{n+1} = y_n + \Delta t\, f\left(t_n + \frac{\Delta t}{2},\ y_n + \frac{1}{2}k\right)$$

Runge-Kutta: RK 4:

$$k_1 = \Delta t\, f(t_n, y_n)$$

$$k_2 = \Delta t\, f\left(t_n + \frac{\Delta t}{2},\ y_n + \frac{1}{2}k_1\right)$$

$$k_3 = \Delta t\, f\left(t_n + \frac{\Delta t}{2},\ y_n + \frac{1}{2}k_2\right)$$

$$k_4 = \Delta t\, f\left(t_n + \Delta t,\ y_n + k_3\right)$$

$$y_{n+1} = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

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Playing with Matlab / Octave:

```
[tm, y1, y2] = harmOsc (0, [1 0], 0.01, 628);
plot (tm, y1)
plot (y1, y2)
```

→ test Runge-Kutta

Set g = 0.9 → no chaos

```
[y1, y2] = rk4pen (0, [2.5 0], 0.01, 628);
```

use 6280, 62800 as well

set g = 1.15 → chaos (see Goldstein, 507)

use `plot (y1, y2, '.', 'MarkerSize', 1);`

(use dt = 0.05, n = 62800)