Hamilton-Jacobi theory

Question: How can we find a transformation to get trivial eqs. of motion?

$$(q_i, p_i) \longrightarrow (Q_i, P_i) \qquad H \longrightarrow K = H + \frac{\partial F}{\partial t}$$

Simplest case: $\dot{Q}_i = \dfrac{\partial K}{\partial P_i} \overset{!}{=} 0$ and $\dot{P}_i = -\dfrac{\partial K}{\partial Q_i} = 0$

So we seek $K(Q_i, P_i, t) = H(q_i, p_i, t) + \dfrac{\partial S}{\partial t} \overset{!}{=} 0$

with $S = F_2$ hence $S = S(q_i, P_i, t)$ and

$$p_i = \frac{\partial S}{\partial q_i}, \quad Q_i = \frac{\partial S}{\partial P_i}$$

$$\Rightarrow \boxed{H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0}$$

Hamilton-Jacobi equation

Example: harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2, \quad p = \frac{\partial S}{\partial q}$$

$$\Rightarrow \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{m}{2}\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

Separation: $S = W(q) + T(t)$

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{m}{2}\omega^2 q^2 + \frac{dT}{dt} = 0 \quad \text{or}$$

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{m}{2}\omega^2 q^2 = -\frac{dT}{dt} = \alpha = \text{const.}$$

$$\Rightarrow T = -\alpha t, \quad \frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{m}{2}\omega^2 q^2 = \alpha$$

$$\Rightarrow \left(\frac{\partial W}{\partial q}\right)^2 = 2m\alpha - m^2\omega^2 q^2 \quad \text{or}$$

$$W = \int \sqrt{2m\alpha - m^2\omega^2 q^2}\, dq$$ Hence we set $\alpha \equiv P$

$$\Rightarrow S(q, P) = \int \sqrt{2mP - m^2\omega^2 q^2}\, dq - Pt$$ then

$$Q = \frac{\partial S}{\partial P} = \int \frac{m\, dq}{\sqrt{2mP - m^2\omega^2 q^2}} - t$$

or $$Q = \int \frac{m\,dq}{\sqrt{2mP}\sqrt{1-\frac{m\omega^2}{2P}q^2}} - t$$

$$= \frac{1}{\omega}\arcsin\left(\sqrt{\frac{m\omega^2}{2P}}\,q\right) - t$$

$$\Rightarrow \quad q = \sqrt{\frac{2P}{m\omega^2}}\,\sin(\omega t + \omega Q),$$

$$p = \frac{\partial S}{\partial q} = \sqrt{2mP - m^2\omega^2 q^2} = \sqrt{2mP}\,\cos(\omega t + \omega Q)$$

General case: $\quad H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0$

complete solution $\rightarrow$ we expect $n+1$ integration constants α_i

Hamilton-Jacobi equation only contains derivatives

$$\Rightarrow \quad S = S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t) + \alpha_{n+1}$$

$\Rightarrow$ set $P_i \equiv \alpha_i$ $\qquad\qquad$ ↑ irrelevant

$\Rightarrow$ $Q_i = \partial S / \partial P_i$

$\Rightarrow$ solve for $q_i = q_i(Q_k, P_k, t)$, find $p_i = \frac{\partial S}{\partial q_i}$ as $p_i = p_i(Q_k, P_k, t)$

IF we can do this → we call the system "integrable".

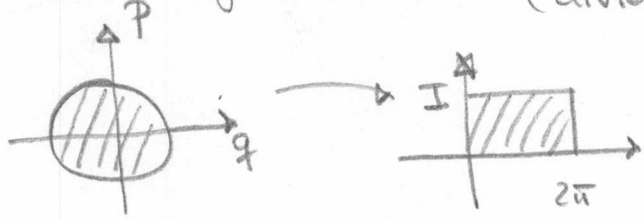
Action-angle variables

Consider $n=1$ and H does not depend explicitly on time:

$$H = \frac{p^2}{2m} + V(q) = E$$

We are looking for a transformation $(q,p) \to (\theta, I)$ such that $\dot{I} = 0$ and $\dot{\theta} = \frac{\partial H}{\partial I} = \omega = \text{const}$. Hence H should only depend on I, or $H(I) = E$.

Idea: Use area of the phase space enclosed by an orbit (divided by 2π)



$$\boxed{I = \frac{1}{2\pi} \oint p \, dq}$$

This works: $\frac{p^2}{2m} = E - V$ or $p = \sqrt{2m}\sqrt{E - V(q)}$

$$p = m\dot{q} = m\frac{dq}{dt} \quad \text{or} \quad dt = m\frac{dq}{p} = \sqrt{\frac{m}{2}} \frac{dq}{\sqrt{E - V(q)}}$$

single orbit: $T = \frac{2\pi}{\omega} = \oint dt = \sqrt{\frac{m}{2}} \oint \frac{dq}{\sqrt{E - V(q)}}$

$$= \frac{d}{dE} \oint \sqrt{2m} \sqrt{E - V(q)}\, dq = \frac{d}{dE} \oint p\, dq =$$

$2\pi \frac{dI}{dE}$ hence $\frac{1}{\omega} = \frac{dI}{dE} \Rightarrow \frac{dE}{dI} = \frac{\partial H}{\partial I} = \omega$

as desired. We can also compute the angle θ:

$$t = \int dt = \frac{d}{dE} \int p\, dq \quad \text{and} \quad \theta = \omega t = \omega \frac{d}{dE} \int p\, dq$$

$$= \omega \frac{dI}{dE} \frac{\partial}{\partial I} \int p\, dq = \frac{\partial}{\partial I} \int p\, dq \quad \text{hence}$$

$$\boxed{\theta = \frac{\partial}{\partial I} \int p\, dq}$$

The two-body problem is integrable :

$$L = \frac{m}{2}\left(\dot r^2 + r^2\dot\varphi^2\right) - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r \;, \quad p_\varphi = \frac{\partial L}{\partial \dot\varphi} = m r^2 \dot\varphi$$

$$H = p_r \dot r + p_\varphi \dot\varphi - L = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + V(r)$$

Hamilton-Jacobi:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 + V(r) + \frac{\partial S}{\partial t} = 0$$

$$S(r,\varphi,\alpha_1,\alpha_2,t) = W_r(r) + W_\varphi(\varphi) - \alpha_1 t \qquad \alpha_1 \equiv E$$

$$\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dW_\varphi}{d\varphi}\right)^2 + V(r) = E$$

$$\frac{1}{2m}\left(\frac{dW_\varphi}{d\varphi}\right)^2 = -\frac{r^2}{2m}\left(\frac{dW_r}{dr}\right)^2 + r^2\left(E - V(r)\right) \quad \text{hence}$$

$$\frac{dW_\varphi}{d\varphi} = \alpha_2 = \text{const} \quad \text{and} \quad \frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + \frac{\alpha_2^2}{2mr^2} = E - V(r)$$

$$\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 = E - V_{eff}(r), \qquad V_{eff} = V(r) + \frac{\alpha_2^2}{2mr^2}$$

$$\frac{dW_r}{dr} = \pm \sqrt{2m\,(E - V_{eff}(r))}$$

$$\Rightarrow \boxed{S = \pm \int^{r} dr' \sqrt{2m\,(E - V_{eff}(r'))} + \alpha_2 \varphi - \alpha_1 t}$$