## Hamiltonian Formalism

Remember: Lagrangian $L = L(q_i, \dot{q}_i, t)$ leads to Euler equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

If $q = (q_1, \ldots, q_s)$ we have $s$ second-order differential equations. This should be equivalent to $2s$ first order equations.

We can use a Legendre-transformation:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{canonical momentum}$$

Form
$$H = H(q_i, p_i, t) = \sum \underbrace{p_i}_{\text{new variable}} \underbrace{\dot{q}_i}_{\text{old variable}} - L \quad (*)$$

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

$$(*) \Rightarrow dH = \sum_i \underbrace{p_i\, d\dot{q}_i}_{\text{killed}} + \dot{q}_i dp_i - \underbrace{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i}_{\text{killed}} - \underbrace{\frac{\partial L}{\partial q_i}}_{(=\dot{p}_i \text{ from Euler eqs})} dq_i - \frac{\partial L}{\partial t} dt + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \boxed{\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = \frac{-\partial H}{\partial q_i} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$ Hamilton's Equations

Remark: If $L = \frac{1}{2} m \dot{q}^2 - V(q) = T - V$

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

$$H = p\dot{q} - L = m\dot{q}^2 - \frac{1}{2} m\dot{q}^2 + V = \frac{1}{2} m\dot{q}^2 + V$$

$$= T + V = \frac{1}{2} \frac{p^2}{m} + V(q)$$

Hamilton's eqs: $\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$ and $\dot{p} = -\frac{\partial H}{\partial q} = -V'(q)$

recover Newton's Law: $\ddot{q} = -\frac{1}{m} V'$

Use this to remember the sign in Hamilton's equations.

Ex: harmonic oscillator: $F = -Dq \quad V = \frac{1}{2} D q^2$

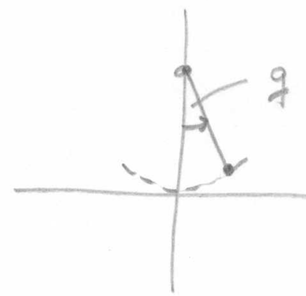
$$L = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} D q^2 \quad H = \frac{p^2}{2m} + \frac{1}{2} D q^2$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \dot{p} = -\frac{\partial H}{\partial q} = -Dq$$

$$\Rightarrow \ddot{q} = -\frac{D}{m} q \Rightarrow \omega = \sqrt{\frac{D}{m}}$$

**Ex:** Mathematical pendulum

$$T = \frac{1}{2} m l^2 \dot{q}^2 \qquad V = mgl(1 - \cos q)$$



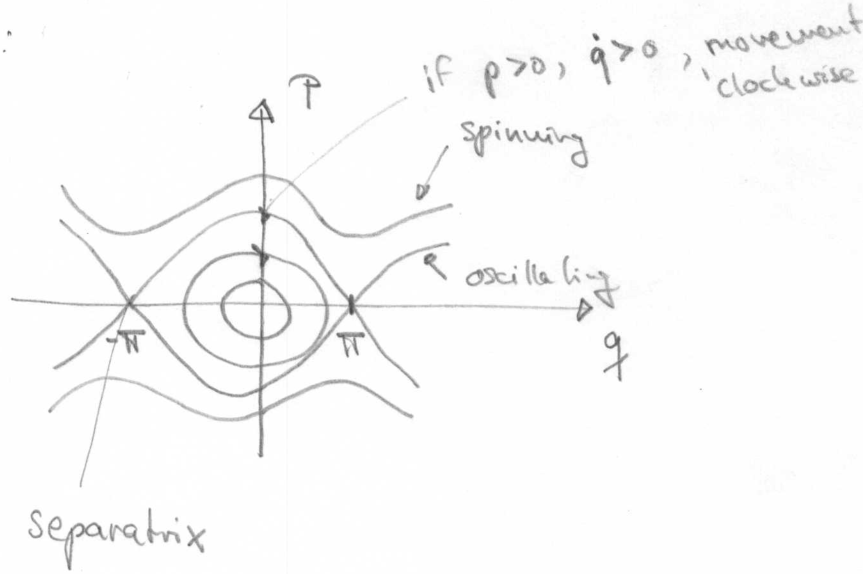

$$\Rightarrow L = \frac{1}{2} m l^2 \dot{q}^2 - mgl(1 - \cos q)$$

$$p = \frac{\partial L}{\partial \dot{q}} = m l^2 \dot{q} \Rightarrow \dot{q} = \frac{p}{m l^2}$$

$$H = T + V = \frac{m}{2} l^2 \frac{p^2}{m^2 l^4} + mgl(1 - \cos q)$$

$$= \frac{p^2}{2 m l^2} + mgl(1 - \cos q)$$

phase portrait:




Using the phase portrait, we can understand dynamics without actually solving the eqs. of motion.

## Liouville's Theorem:

Consider a small volume in phase space. The shape of the volume will gradually change, but the volume remains the same

**Proof:**

$dV = dq \cdot dp$ for small area. For small time step $dt$, we have

$$q \to q + \dot{q}\,dt = q + \frac{\partial H}{\partial p} dt = \tilde{q}, \qquad p \to p + \dot{p}\,dt = p - \frac{\partial H}{\partial q} dt = \tilde{p}$$

$$d\tilde{V} = d\tilde{q} \cdot d\tilde{p} = (\det J)\, dV \quad \text{where}$$

$$J = \begin{pmatrix} \partial\tilde{q}/\partial q & \partial\tilde{q}/\partial p \\ \partial\tilde{p}/\partial q & \partial\tilde{p}/\partial p \end{pmatrix}$$

and we show $\det J = 1 + \mathcal{O}(dt^2)$

$$\det J = \begin{vmatrix} 1 + \frac{\partial^2 H}{\partial q \partial p} dt & \frac{\partial^2 H}{\partial p^2} dt \\ -\frac{\partial^2 H}{\partial q^2} dt & 1 - \frac{\partial^2 H}{\partial q \partial p} dt \end{vmatrix} = 1 + \mathcal{O}(dt)$$

Note: We can also introduce $\varrho$ as density in phase space

$$\varrho = \varrho(q, p, t) = \frac{\Delta N}{\Delta V}$$

$\Delta N$ — number of particles in $\Delta V$

$\Delta V$ — volume of $\Delta V$

assuming no particle is created or distroyed: $\frac{d\varrho}{dt} = 0$

or

$$\frac{\partial \varrho}{\partial t} + \sum_i \frac{\partial \varrho}{\partial q_i} \dot{q}_i + \frac{\partial \varrho}{\partial p_i} \dot{p}_i = 0$$

$$\frac{\partial \varrho}{\partial t} + \sum_i \frac{\partial \varrho}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \varrho}{\partial p_i}\frac{\partial H}{\partial q_i} = 0$$

$$\{A, B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$ Poisson - brackets

$$\frac{\partial \varrho}{\partial t} + \{\varrho, H\} = 0$$ (Liouville equation)

Remark: Poincare Recurrence Theorem: (bounded phase space)
Consider a point P in the phase space. Then for any neighborhood $D_0$ of P, there exists a point $P' \in D_0$ that will return to $D_0$ in a finite time

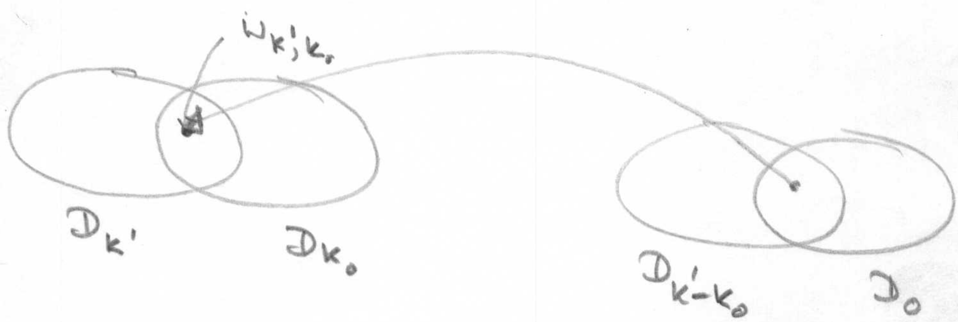
Consider the evolution of $D_0$ over a finite time interval $T$. Hamilton's equations: map $D_0 \to D_1$

$Vol(D_0) = Vol(D_1)$, shapes are different.

Set $D_k$ the region after time $k \cdot T$ ($k$ is an integer)

Then, there exists a $k_0$ such that $D_{k_0} \cap D_{k'} \neq \emptyset$

Assume $k' > k_0$, $\omega_{k';k_0} \in D_{k_0} \cap D_{k'}$




$D_{k'}$ resulted from $D_0$ in $k'$ steps

$D_{k_0}$ resulted from $D_0$ in $k_0$ steps

$\Rightarrow$ $D_{k'-k_0} \cap D_0 \neq \emptyset$

$\blacksquare$

Transformations that leave the canonical equations invariant are called <u>canonical transformations</u>

$$(q_i, p_i) \longrightarrow (Q_i, P_i) \quad \text{such that}$$
$$H \longrightarrow K \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}, \quad \dot{Q}_i = \frac{\partial K}{\partial P_i}$$

How can we find canonical transformations?
We know that they are derived from a variational principle:

$$\delta \int_{t_0}^{t_1} \left( \sum p_i \dot{q}_i - H \right) dt = 0 \quad ; \quad \delta \int_{t_0}^{t_1} \left( \sum P_i \dot{Q}_i - K \right) dt = 0$$

Therefore, if integrands only differ by a total derivative in time $\frac{dF}{dt}$, both systems will describe the same dynamics.

$$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \frac{dF}{dt}$$

The function F is called <u>"generating" function</u>.

One way to construct canonical transformations is to consider $F$ to depend on 'old' and 'new' coordinates:

$$F_1 = F_1(q_i, Q_i, t) \qquad F_3 = F_3(p_i, Q_i, t)$$
$$F_2 = F_2(q_i, P_i, t) \qquad F_4 = F_4(p_i, P_i, t)$$

Examples: (a) $F = F_1(q_i, Q_i, t)$

$$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \sum \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t}$$

$$\Rightarrow \quad p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}, \quad K = H + \frac{\partial F_1}{\partial t}$$

(b) $F = F_2(q_i, P_i, t)$

$$\sum p_i \dot{q}_i - H = \underbrace{\sum P_i \dot{Q}_i}_{\frac{d}{dt}\sum(P_i Q_i) - \dot{P}_i Q_i} - K + \sum \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i + \frac{\partial F_2}{\partial t}$$

$$\Rightarrow \quad p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i}, \quad K = H + \frac{\partial F_2}{\partial t}$$

**Example:** Harmonic oscillator

$$H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

$$F_1(q,Q) = \frac{m\omega}{2} q^2 \cot Q$$

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \qquad P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega}{2} q^2 \frac{1}{\sin^2 Q}$$

$$p^2 = m^2\omega^2 q^2 \cot^2 Q \quad , \quad q^2 = \frac{2P}{m\omega} \sin^2 Q$$

$$\Rightarrow p^2 = m^2\omega^2 \frac{2P}{m\omega} \cos^2 Q = 2P m\omega \cos^2 Q$$

$$\Rightarrow H(q,p) = K(Q,P) = \frac{2P m\omega \cos^2 Q}{2m} + \frac{m\omega^2}{2} \cdot \frac{2P}{m\omega} \sin^2 Q$$

$$= \omega P$$

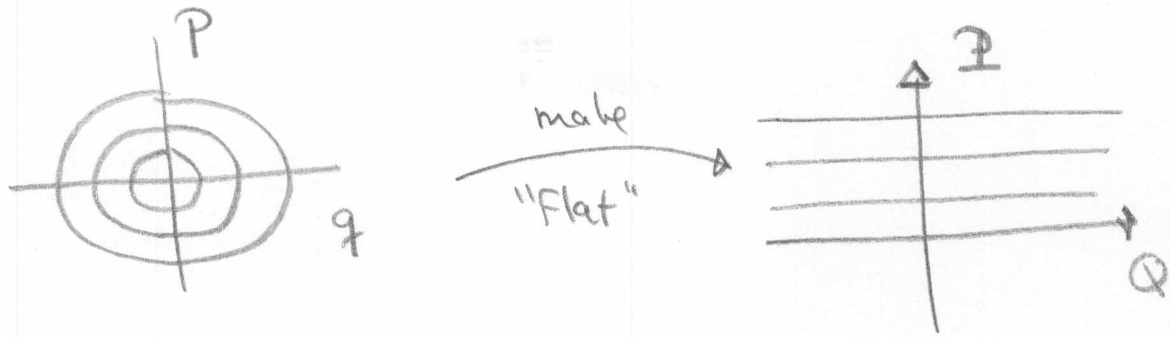
$$\Rightarrow \dot{Q} = \frac{\partial K}{\partial P} = \omega \quad \text{and} \quad \dot{P} = -\frac{\partial K}{\partial Q} = 0$$

(Eqs of motion are trivial).

$$\Rightarrow Q = \omega t + \alpha \quad , \quad P = \text{const.} = \frac{E}{\omega}$$

$$\Rightarrow q = \sqrt{\frac{2P}{m\omega}} \sin Q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha)$$

$$p = \sqrt{2P m\omega} \cos Q = \sqrt{2mE} \cos(\omega t + \alpha)$$



phase space $(q, p)$ plase space $(Q, P)$

We will later discuss methods to find canonical transformations that "Flatten" the phase space.

How can we decide whether a transformation is canonical without looking for a generating Function?

Answer: Invariance of Poisson-brackets.

To see this, let's look at the symplectic structure of the phase space:

We can write Hamilton's eqs as

$$\dot{x} = J \frac{\partial H}{\partial x} \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

and $x = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix}$, $\frac{\partial H}{\partial x} = \nabla_x H$

A canonical transform shall preserve this structure

Let $x \to y$ correspond to $(q,p) \to (Q,P)$.

Then $\dot{y}_i = \frac{\partial y_i}{\partial x_k}\dot{x}_k = \frac{\partial y_i}{\partial x_k} J_{k\ell}\frac{\partial H}{\partial x_\ell} = \frac{\partial y_i}{\partial x_k} J_{k\ell}\frac{\partial y_m}{\partial x_\ell}\frac{\partial H}{\partial y_m}$

(sum convention!)

or, in matrix notation, $M = \left(\partial y_i / \partial x_k\right)$ is the Jacobian

$$\dot{y} = M J M^T \frac{\partial H}{\partial y}$$

Thus, for a canonical transform: $\boxed{M J M^T = J}$

Let's revise the Poisson-brackets:

$$\{A, B\}_x = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} = (\partial_x A)\cdot J\,(\partial_x B)$$

$$= \begin{pmatrix} \partial A/\partial q_1 \\ \vdots \\ \partial A/\partial q_n \\ \partial A/\partial p_1 \\ \vdots \\ \partial A/\partial p_n \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial B/\partial q_1 \\ \vdots \\ \partial B/\partial q_n \\ \partial B/\partial p_1 \\ \vdots \\ \partial B/\partial p_n \end{pmatrix} = \frac{\partial A}{\partial x_m} J_{mk} \frac{\partial B}{\partial x_k}$$

$$= \frac{\partial A}{\partial y_\ell}\frac{\partial y_\ell}{\partial x_m} J_{mk} \frac{\partial y_i}{\partial x_k}\frac{\partial B}{\partial y_i} = (\partial_y A)\cdot M J M^T (\partial_y B)$$

Hence $\{A, B\}_x = (\partial_y A) \cdot M J M^T (\partial_y B)$

$= (\partial_y A) \cdot J (\partial_y B) = \{A, B\}_y$ if $J = M J M^T$

(or if the transformation $x \to y$ is canonical)

Hence we have shown: transformation is canonical

$\Rightarrow$ Poisson-brackets are invariant.

We can also show "$\Leftarrow$". Here $n=1$ (general case: exercise)

Idea: Show $M J M^T = J$:

$$\begin{pmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial p \end{pmatrix} J \begin{pmatrix} \partial Q/\partial q & \partial P/\partial q \\ \partial Q/\partial p & \partial P/\partial p \end{pmatrix}$$

$$= \begin{pmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial Q/\partial q & \partial P/\partial q \\ \partial Q/\partial p & \partial P/\partial p \end{pmatrix}$$

$$= \begin{pmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial p \end{pmatrix} \begin{pmatrix} \partial Q/\partial p & \partial P/\partial p \\ -\partial Q/\partial q & -\partial P/\partial q \end{pmatrix} = \begin{pmatrix} \{Q, Q\}_x & \{Q, P\}_x \\ \{P, Q\}_x & \{P, P\}_x \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ if P.B. are invariant.

Example: $P = \frac{m\omega}{2} q^2 \frac{1}{\sin^2 Q}$ (see above)

$$p = \sqrt{2Pm\omega}\,\cos Q \qquad q = \sqrt{\frac{2P}{m\omega}}\,\sin Q$$

$$\partial p/\partial P = \frac{\cancel{2}m\omega}{\cancel{2}\sqrt{2Pm\omega}}\cos Q \qquad \frac{\partial q}{\partial P} = \frac{\cancel{2}/m\omega}{\cancel{2}\sqrt{\frac{2P}{m\omega}}}\sin Q$$

$$\partial p/\partial Q = -\sqrt{2Pm\omega}\,\sin Q \qquad \frac{\partial q}{\partial Q} = \sqrt{\frac{2P}{m\omega}}\,\cos Q$$

$$\{q,p\}_y = \frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial p}{\partial Q} = \frac{m\omega}{\sqrt{2Pm\omega}}\sqrt{\frac{2P}{m\omega}}\cos^2 Q$$

$$+ \frac{1}{\sqrt{2Pm\omega}}\sqrt{2Pm\omega}\,\sin^2 Q = 1$$

which is sufficient to show the invariance of the Poisson-brackets.