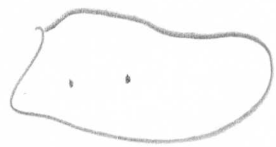
Analytical Dynamics

Lecture 8

Remember:

Rigid body:  distances between points are fixed.

How many coordinates do we need to describe its motion?

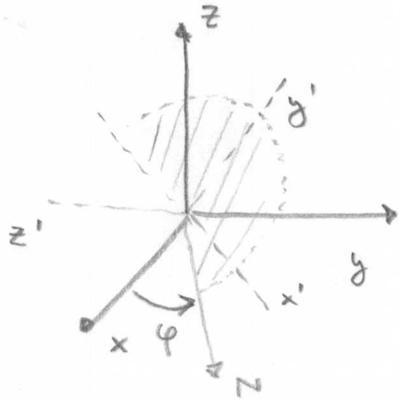
Answer: 6

Choose a point (3 coordinates), fix an axis (direction)

Euler Angles { (two coordinates) and rotate about this axis a certain angle (one coordinate)

Rotations in space

Imagine we have an (x, y, z) Cartesian system and also an (x', y', z') Cartesian system. Which steps can we take to go from (x, y, z) to (x', y', z')?



Idea: The (x', y') plane intersects with the (x, y) plane in N

So we can first rotate around the z-axis such that the x-axis coincides with N.

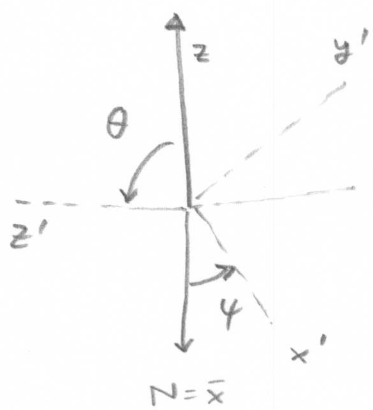
This angle is called φ.

$$\bar{x} = x\cos\varphi + y\sin\varphi$$
$$\bar{y} = -x\sin\varphi + y\cos\varphi$$
$$\bar{z} = z$$

$$D = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Next step:

Rotate around N ($\bar{x}$-axis) such that z coincides with z'.



$$\tilde{x} = \bar{x}$$
$$\tilde{y} = \bar{y}\cos\theta + \bar{z}\sin\theta$$
$$\tilde{z} = -\bar{y}\sin\theta + \bar{z}\cos\theta$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

Last step: Rotate around $\tilde{z}$ axis such that $\bar{x}$ axis coincides with x' axis.

$$x' = \tilde{x}\cos\psi + \tilde{y}\sin\psi$$
$$y' = -\tilde{x}\sin\psi + \tilde{y}\cos\psi$$
$$z' = \tilde{z}$$

$$B = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Entire transformation:

$$A = BCD =$$

$$\begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix}$$

$$\vec{r}\,' = A\vec{r}$$

Infinetesimal rotations:

Consider "small" $d\varphi, d\theta, d\psi$:

$$A_{inf} = \begin{pmatrix} 1 & d\varphi + d\psi & 0 \\ -d\psi - d\varphi & 1 & d\theta \\ 0 & -d\theta & 1 \end{pmatrix} = 1 + \begin{pmatrix} 0 & d\omega_3 & 0 \\ -d\omega_3 & 0 & d\omega_1 \\ 0 & -d\omega_1 & 0 \end{pmatrix}$$

with $d\vec{\Omega} = \begin{pmatrix} d\theta \\ 0 \\ d\phi + d\psi \end{pmatrix}$

Angular velocity in Euler angles.

(we could use $\tilde{\omega}_{ij} = \frac{dR_{ik}}{dt} R^{-1}_{kj}$ as we know R explicitly)

Consider motion for small dt:

$$(\psi, \theta, \varphi) \to (\psi + d\psi, \theta + d\theta, \phi + d\theta)$$

$$\Rightarrow \quad \omega = \dot{\phi}\, \bar{e}_3 + \dot{\theta}\, \underset{\text{N-axis}}{\tilde{e}_1} + \dot{\psi}\, e_3$$

and

$$\bar{e}_3 = \sin\theta \sin\psi\, e_1 + \sin\theta \cos\psi\, e_2 + \cos\theta\, e_3$$

$$\tilde{e}_1 = \cos\psi\, e_1 - \sin\psi\, e_2$$

Proof: $\tilde{e}_1 = B^{-1}_{1k} e_k = \cos\psi\, e_1 - \sin\psi\, e_2$ and

$$\bar{A}^{-1} = \begin{pmatrix} * & * & * \\ * & * & * \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

$$\Rightarrow \quad w = (\dot{\phi} \sin\theta \sin\psi + \dot{\theta}\cos\psi)\, e_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\, e_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\, e_3$$

$\dot{\psi}$: spinning about its own figure axis

$\dot{\phi}$: precession of the figure axis about the vertical axis

$\dot{\theta}$: nutation (bobbing up and down) of the figure axis relative to the vertical space axis