Analytical Dynamics

Lecture 7

Rigid Body Motion

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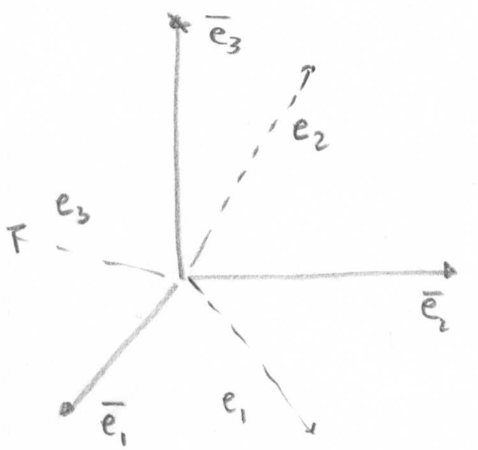
Rigid body: distance between points is fixed

6 degrees of freedom: 3 translation + 3 rotation.

Consider a body fixed at a point $\to$ rotation



$(\bar{e}_1, \bar{e}_2, \bar{e}_3) \longrightarrow$ fixed space frame

$(e_1(t), e_2(t), e_3(t)) \longrightarrow$ moving body frame

Both coordinate systems are orthogonal: $\bar{e}_i \cdot \bar{e}_j = \delta_{ij}$

$e_i \cdot e_j = \delta_{ij}$

$\Rightarrow$ $\exists$ an orthogonal matrix $R = (R_{ik})$ such that

$$(*) \qquad e_i = R_{ik}(t)\, \bar{e}_k = \sum_{k=1}^{3} R_{ik}(t)\, \bar{e}_k$$

Proof: $e_i \cdot e_j = \delta_{ij} = (R_{ik}\, \bar{e}_k) \cdot (R_{jm}\, \bar{e}_m) = R_{ik} R_{jm} \overbrace{\bar{e}_k \cdot \bar{e}_m}^{\delta_{km}}$

$= R_{ik} R_{jk} = (R^T R)_{ij} \Rightarrow R$ is orthogonal

From $(*)$: $R_{ik} = e_i \cdot \bar{e}_k$.

We are now ready to introduce angular velocity:

[We can express any point $\vec{r}$ in space frame or body frame: $r = \bar{r}_i(t)\,\bar{e}_i = r_K\, e_K(t)$ and $\bar{r}_K(t) = r_i\, R_{ik}(t)$

$$= r_K R_{Ki}\, \bar{e}_i$$

$$\frac{dr}{dt} = \frac{d\bar{r}_i}{dt}\,\bar{e}_i = r_i \frac{de_i}{dt} = r_i \frac{dR_{ik}}{dt}\,\bar{e}_k \quad]$$

For the (e_1, e_2, e_3), the body frame, in particular

$$\frac{de_i}{dt} = \frac{dR_{ik}}{dt}\,\bar{e}_k = \frac{dR_{ik}}{dt}\,R^{-1}_{kj}\, e_j \quad \text{as} \quad \bar{e}_k = R^{-1}_{kj}\, e_j$$

$$=: \tilde{\omega}_{ij}\, e_j$$

with $\tilde{\omega}_{ij} = \dot{R}_{ik}(R^{-1})_{kj} = \dot{R}_{ik} R_{jk}$ (as $R^T = R^{-1}$)

It is easy to see that ω is antisymmetric: $\tilde{\omega}_{ik} = -\tilde{\omega}_{ki}$

<u>Proof:</u> $R_{ik} R_{jk} = \delta_{ij} \Rightarrow \dot{R}_{ik} R_{jk} + R_{ik} \dot{R}_{jk} = 0$

$$\Rightarrow \tilde{\omega}_{ij} + \tilde{\omega}_{ji} = 0 \quad \blacksquare$$

$$\tilde{\omega} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

and we can write with $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$

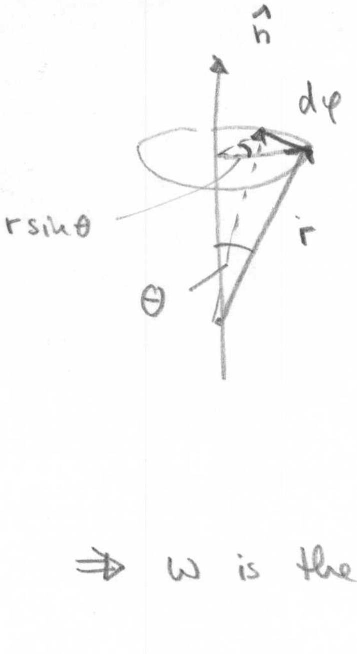
meaning $\omega = \omega_i e_i$

$$\Rightarrow \boxed{\frac{de_i}{dt} = \omega \times e_i}$$

Ex:
$$\frac{de_1}{dt} = \tilde{\omega}_{1j} e_j = \tilde{\omega}_{12} e_2 + \tilde{\omega}_{13} e_3 = \omega_3 e_2 - \omega_2 e_3$$

$$\omega \times e_1 = (\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3) \times e_1 = -\omega_2 e_3 + \omega_3 e_2$$

Interpretation



consider small rotation $d\varphi$ around axis $\hat{n}$

$$|dr| = |r||d\varphi| \sin\theta$$

$$\Rightarrow dr = d\varphi \times r \qquad d\varphi = \hat{n}\,|d\varphi|$$

$$\Rightarrow \frac{dr}{dt} = \omega \times r \quad \text{with} \quad \omega \equiv \frac{d\varphi}{dt}$$

$\Rightarrow$ ω is the instantaneous angular velocity.

We can now express e.g. T the kinetic energy using ω:

$$T = \frac{1}{2}\sum_j m_j \dot{r}_j^2 = \frac{1}{2}\sum_j m_j (\omega \times r_j)\cdot(\omega \times r_j)$$

$$= \frac{1}{2}\sum_j m_j \left((\omega\cdot\omega)(r_j \cdot r_j) - (r_j\cdot\omega)^2 \right)$$

(*) $$= \frac{1}{2}\omega_i I_{ik}\omega_k$$ where <u>the inertia tensor</u> I

is defined as $$I_{ik} = \sum_j m_j \left((r_j\cdot r_j)\delta_{ik} - (r_j)_i (r_j)_k \right)$$

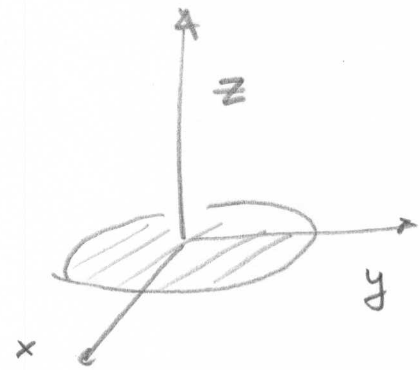
Let the rigid body have a density $\varrho(r)$, then

$$I = \int d^3r\, \varrho(r) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$$

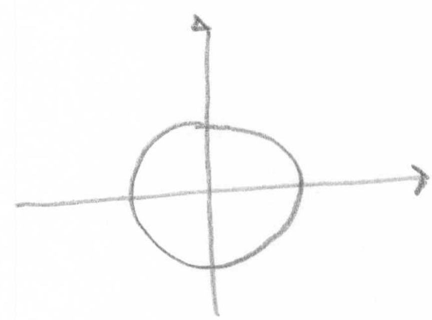
Note: I is symmetric, hence it can be diagonalized: $\exists$ an orthogonal matrix M such that $I' = MIM^T$ is diagonal. The axes which are aligned with the eigenvectors of I are called <u>principal axes</u>

Example: disc: (no mass in z-direction)

$$\varrho = \frac{M}{\pi R^2}$$



$$I_{11} = \int_{\text{Disc}} y^2 \frac{M}{\pi R^2} \, dx\,dy$$



To compute this integral:

$$I_{22} = \int_{\text{Disc}} x^2 \frac{M}{\pi R^2} \, dx\,dy$$

$$I_{33} = I_{11} + I_{22} = \int_{\text{Disc}} x^2 + y^2 \frac{M}{\pi R^2} \, dx\,dy = \frac{M}{\pi R^2} \int_0^R r^3 \, dr \cdot 2\pi$$

$$= 2\pi \frac{M}{\pi R^2} \cdot \frac{R^4}{4} = \frac{MR^2}{2}$$

$$\Rightarrow I_{11} = I_{22} = \frac{MR^2}{4}$$

For the angular momentum:

$$L = \sum_j m_j (r_j \times v_j) = \sum_j m_j (r_j \times (\omega \times r_j))$$

$$= \sum_j m_j [\omega r_j^2 - (r_j \cdot \omega) r_j]$$

e.g. $$L_x = \sum_j m_j [\omega_x r_j^2 - (r_j \cdot \omega) x_j]$$

$$= \sum_j m_j [\omega_x (x_j^2 + y_j^2 + z_j^2) - (x_j \omega_x + y_j \omega_y + z_j \omega_z) x_j]$$

$$= I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$$

$$I_{xx} = \sum_j m_j (y_j^2 + z_j^2) \qquad I_{xy} = \sum_j m_j (-x_j y_j)$$

$$I_{xz} = \sum_j m_j (-x_j z_j)$$

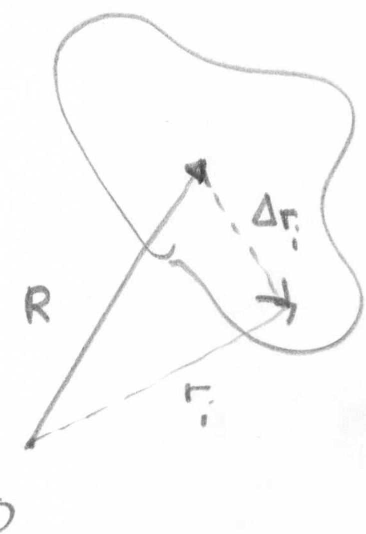
or, in general $\boxed{L = I \omega}$

Analytical Dynamics

Most general motion: overall translation superposed with a rotation

$\hookrightarrow$ useful: consider rotation around center of mass

$$r_j(t) = R(t) + \Delta r_j(t)$$



For center of mass:

$$\sum_j m_j \Delta r_j = 0$$

$\Rightarrow$

$\searrow 0$

$$T = \frac{1}{2} \sum_j m_j \dot{r}_j^2 \qquad (\dot{\Delta r_j} = \omega \times \Delta r_j)$$

$$= \sum_j m_j \left(\frac{1}{2} \dot{R}^2 + \dot{R} \cdot (\omega \times \Delta r_j) + \frac{1}{2} (\omega \times \Delta r_j)^2 \right)$$

$$= \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \omega_i I_{ij} \omega_j$$

(Center of mass: $R = \frac{1}{M} \sum m_j r_j$

$$\Rightarrow \sum_j m_j \Delta r_i = \sum_j m_j (r_j - R) = \sum_j m_j r_j - \sum_j R m_j = 0 \,)$$

Free motion of rigid body

$$\frac{dL}{dt} = 0 \quad \text{and} \quad L = L_k e_k$$

body axes

$$0 = \frac{dL_k}{dt} e_k + L_k \frac{de_k}{dt} = \frac{dL_k}{dt} e_k + L_k (\omega \times e_k) \quad (*)$$

Let body axes coincide with principal axes:

$$L_k = I_k \omega_k \ , \quad \text{so e.g.} \quad L_1 = I_1 \omega_1$$

$$\omega = \omega_j e_j$$

Project $(*)$ on e_i

$$-\frac{dL_i}{dt} = L_k (\omega_j e_j \times e_k) \cdot e_i$$

if $i=1$, we have

$$e_2 \times e_3 = e_1$$

$$e_3 \times e_2 = -e_1$$

$$-\frac{dL_i}{dt} = I_k \omega_k \omega_j \, (e_j \times e_k) \cdot e_i$$

$$0 = I_1 \dot{\omega}_1 + I_3 \omega_3 \omega_2 - I_2 \omega_2 \omega_3$$

$$0 = I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2)$$

⇒ Euler's Equations

$$\boxed{\begin{aligned} I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) &= 0 \\ I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) &= 0 \\ I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) &= 0 \end{aligned}}$$

Example: Symmetric top

$$I_1 = I_2 \neq I_3$$

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_1 - I_3)$$

$$I_2 \dot{\omega}_2 = -\omega_1 \omega_3 (I_1 - I_3)$$

$$I_3 \dot{\omega}_3 = 0$$

⇒ spin about the symmetric axis with constant ω_3

⇒ $\dot{\omega}_1 = \Omega \omega_2$, $\dot{\omega}_2 = -\Omega \omega_1$ $\qquad \Omega = \dfrac{\omega_3 (I_1 - I_3)}{I_1}$

⇒ $(\omega_1, \omega_2) = \omega_0 (\sin \Omega t, \cos \Omega t)$

(Precession about the e_3 axis with frequency Ω)

("wobble" of earth tricky to explain - more complicated)

Example: Stability of asymmetric top:

Assume $I_1 \neq I_2 \neq I_3 \neq I_1$. Spin completely about one of the principal axes:

$$\omega_1 = \Omega, \qquad \omega_2 = \omega_3 = 0 \qquad \text{solves Euler's eqs.}$$

What happens to a small perturbation:

$$\omega_1 = \Omega + \eta_1, \quad \omega_2 = \eta_2, \quad \omega_3 = \eta_3$$

$$\Rightarrow \quad I_1 \dot{\eta}_1 = 0$$

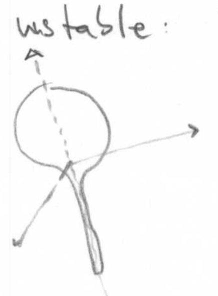
$$I_2 \dot{\eta}_2 = \Omega \eta_3 (I_3 - I_1)$$

$$I_3 \dot{\eta}_3 = \Omega \eta_2 (I_1 - I_2)$$

$$\Rightarrow \quad I_2 \ddot{\eta}_2 = \underbrace{\frac{\Omega^2}{I_3}(I_3 - I_1)(I_1 - I_2)}_{\alpha} \eta_2$$

if $\alpha > 0 \Rightarrow$ unstable, $\quad \alpha < 0 \Rightarrow$ stable

unstable: $I_2 < I_1 < I_3$ or $I_3 < I_1 < I_2$



stable about axis with smallest and largest moment of inertia.