Lecture 6



## Small oscillations:

Consider $x_i = x_i(q_1, \dots, q_s)$ , $T = \frac{1}{2} \sum_i m_i \dot{x}_i^2$

Assume that there is an equilibrium at $(q_{10}, \dots, q_{s0})$.

Then we introduce $\xi_j = q_j - q_{j0}$ as deviations from the equilibrium. Obviously (as $\dot{q}_{j0} = 0$) $\dot{\xi}_j = \dot{q}_j$,

$\dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j} \dot{q}_j$ , $\xi_j$ is assumed to be small

$$\Rightarrow \quad T = \frac{1}{2} \sum_{n,m=1}^{s} \mu_{nm} \dot{\xi}_n \dot{\xi}_m \qquad \mu_{nm} = \sum_i m_i \frac{\partial x_i}{\partial q_n} \frac{\partial x_i}{\partial q_m} \Big|_{q=q_0}$$

$\mu$ is symmetric and positive definite

Stable equilibrium: $U = U(q)$ has a minimum at $q_0$

hence $\frac{\partial U}{\partial q_j}\Big|_{q=q_0} = 0$

$$\Rightarrow \quad U = U(q_0) + \frac{1}{2} \sum_{n,m}^{s} K_{nm} \xi_n \xi_m , \quad K_{nm} = \frac{\partial^2 U}{\partial q_n \partial q_m} \Big|_{q=q_0}$$

$$\Rightarrow \quad \boxed{L = \frac{1}{2} \sum_{n,m=1}^{s} \mu_{nm} \dot{\xi}_n \dot{\xi}_m - K_{nm} \xi_n \xi_m}$$

$$\frac{\partial L}{\partial \dot{\xi}_n} = \sum_{m=1}^{s} \mu_{nm} \dot{\xi}_m \qquad \frac{\partial L}{\partial \xi_n} = -\sum_{m=1}^{s} k_{nm} \xi_m$$

$$\Rightarrow \sum_{m=1}^{s} \mu_{nm} \ddot{\xi}_m + k_{nm} \xi_m = 0$$

Set $\xi_m = a_m e^{-i\omega t}$ $\Rightarrow$ $\underbrace{(K - \omega^2 \mu)}_{\text{matrix}} \begin{pmatrix} a_1 \\ \vdots \\ a_s \end{pmatrix} = 0$

$\Rightarrow$ solve $|K - \omega^2 \mu| = 0$ to obtain $\lambda_k = \omega_k^2$

The $\omega_k = \sqrt{\omega_k^2}$ are called <u>eigenfrequencies</u> of the problem.

If we choose $a^{(k)}$ such that $K a^{(k)} = \lambda_k \mu a^{(k)}$,

then the $\xi^{(k)} = a^{(k)} e^{-i\omega_k t}$ are called <u>normal modes</u>

$\xi = \sum_k a^{(k)} B_k \cos(\omega_k t + \varphi_k)$ is the general solution

Note: $\omega_k^2 = 0$ leads to the term $\xi_0 + \dot{\xi}_0 t$ (translation)

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<u>Remark</u>: Product of two symmetric matrices is not necessarily symmetric.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

**Remark:** $\lambda_k = \omega_k^2$ is real. To show this consider

$$K a^{(k)} = \lambda_k \mu a^{(k)}$$

and multiply both sides by $\bar{a}^{(k)}$

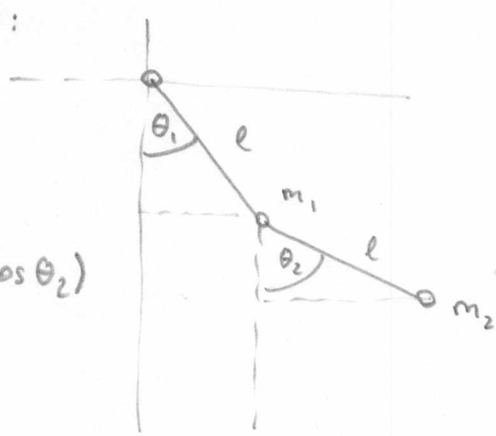
$$\underbrace{\bar{a}^{(k)} \cdot K a^{(k)}}_{\in \mathbb{R}} = \lambda_k \underbrace{\bar{a}^{(k)} \cdot \mu a^{(k)}}_{\in \mathbb{R}}$$

(remember, if $M$ is symmetric, it has a complete basis $(\varphi_k)$ $\Rightarrow$ $a = \sum a_k \varphi_k$, $\bar{a} = \sum \bar{a}_k \varphi_k$,
$\bar{a} \cdot M a = \sum_{i,j} \bar{a}_i a_j \lambda_j \varphi_i \cdot \varphi_j \in \mathbb{R}$)

**Example:** Double pendulum

$$L = m l^2 \dot{\theta}_1^2 + \frac{1}{2} m l^2 \dot{\theta}_2^2 + m l^2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2$$
$$+ 2 m g l \cos\theta_1 + m g l \cos\theta_2$$

To see this:




$m = m_1 = m_2$

$$y_1 = -l\cos\theta_1$$
$$y_2 = -(l\cos\theta_1 + l\cos\theta_2)$$
$$T_1 = \frac{1}{2} m l^2 \dot{\theta}_1^2$$

$T_2$ requires a little bit of work.

$$\vec{r}_2 = \begin{pmatrix} l \sin\theta_1 + l \sin\theta_2 \\ -l\cos\theta_1 - l\cos\theta_2 \end{pmatrix}$$

$$\dot{\vec{r}}_2 = l \begin{pmatrix} \dot\theta_1 \cos\theta_1 + \dot\theta_2 \cos\theta_2 \\ \dot\theta_1 \sin\theta_1 + \dot\theta_2 \sin\theta_2 \end{pmatrix}$$

$$\dot{r}^2 = l^2 \left[ \dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) \right]$$

Now we linearize $L$ and obtain (up to 2<sup>nd</sup> order)

$$L \approx ml^2 \dot\theta_1^2 + \frac{1}{2} ml^2 \dot\theta_2^2 + ml^2 \dot\theta_1 \dot\theta_2 - mgl\,\theta_1^2 - \frac{1}{2} mgl\,\theta_2^2$$

Euler-Lagrange:

$$2ml^2 \ddot\theta_1 + ml^2 \ddot\theta_2 = -2mgl\,\theta_1$$
$$ml^2 \ddot\theta_2 + ml^2 \ddot\theta_1 = -mgl\,\theta_2$$

$$2\ddot\theta_1 + \ddot\theta_2 = -\frac{2g}{l}\theta_1$$
$$\ddot\theta_2 + \ddot\theta_1 = -\frac{g}{l}\theta_2$$

Set $\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-i\omega t}$

$$\Rightarrow \quad -2\omega^2 a_1 - \omega^2 a_2 = -\frac{2g}{\ell} a_1$$
$$-\omega^2 a_1 - \omega^2 a_2 = -\frac{g}{\ell} a_2$$

$$\Rightarrow (*) \begin{cases} \left(-2\omega^2 + \frac{2g}{\ell}\right) a_1 - \omega^2 a_2 = 0 \\ -\omega^2 a_1 + \left(-\omega^2 + \frac{g}{\ell}\right) a_2 = 0 \end{cases}$$

$M a = 0$

$|M| = 0$

$$\Rightarrow \left(\frac{2g}{\ell} - 2\omega^2\right)\left(\frac{g}{\ell} - \omega^2\right) - \omega^4 = 0$$

$$\frac{2g^2}{\ell^2} - 4\omega^2 \frac{g}{\ell} + \omega^4 = 0$$

$$\omega^4 - 4\omega^2 \frac{g}{\ell} + 4\left(\frac{g}{\ell}\right)^2 = 2\left(\frac{g}{\ell}\right)^2$$

$$\omega^2 - 2\frac{g}{\ell} = \pm \frac{g}{\ell}\sqrt{2}$$

$$\Rightarrow \quad \omega_1^2 = (2+\sqrt{2})\frac{g}{\ell}, \qquad \omega_2^2 = (2-\sqrt{2})\frac{g}{\ell}$$

$\omega_2$ :

$\omega_1$

$\omega_1 > \omega_2$

higher frequency

Normal vectors :

① $\omega^2 = \omega_1^2 = (2+\sqrt{2})\frac{g}{\ell}$ in (*)

$$\left(\left(-4-2\sqrt{2}\right)\frac{g}{\ell} + \frac{2g}{\ell}\right)a_1 - (2+\sqrt{2})\frac{g}{\ell}a_2 = 0$$

$$-(2+\sqrt{2})\frac{g}{\ell}a_1 + \left(-(2+\sqrt{2})\frac{g}{\ell} + \frac{g}{\ell}\right)a_2 = 0$$

$$-(2+2\sqrt{2})a_1 - (2+\sqrt{2})a_2 = 0 \Rightarrow a_2 = \frac{-(2+2\sqrt{2})}{2+\sqrt{2}}a_1$$

$$-(2+\sqrt{2})a_1 + (-\sqrt{2}-1)a_2 = 0 \Rightarrow a_2 = -\frac{2+\sqrt{2}}{1+\sqrt{2}}a_1$$

$$\frac{2+\sqrt{2}}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{\cancel{2}-2\sqrt{2}+\sqrt{2}\cancel{-2}}{1-2} = \sqrt{2}$$

$$\Rightarrow \vec{a} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

② $\omega^2 = \omega_2^2 = (2-\sqrt{2})\frac{g}{\ell}$

$$\left((-4+2\sqrt{2})+2\right)a_1 - (2-\sqrt{2})a_2 = 0$$

$$a_2 = \frac{2\sqrt{2}-2}{2-\sqrt{2}}a_1 = \frac{(2\sqrt{2}-2)(2+\sqrt{2})}{4-2}a_1$$

$$= \frac{4\sqrt{2}+\cancel{4}-\cancel{4}-2\sqrt{2}}{2}a_1 = \sqrt{2}\,a_1$$

$$\Rightarrow \vec{a} = \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix}$$

The general solution is given by

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \alpha_1 \cos(\omega_1 t + \varphi_1)\begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix} + \alpha_2 \cos(\omega_2 t + \varphi_2)\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$$

as a superposition of both modes:

$$\theta_1 = \alpha_1 \cos(\omega_1 t + \varphi_1) + \alpha_2 \cos(\omega_2 t + \varphi_2)$$

$$\theta_2 = -\sqrt{2}\,\alpha_1 \cos(\omega_1 t + \varphi_1) + \sqrt{2}\,\alpha_2 \cos(\omega_2 t + \varphi_2)$$

$$= \begin{pmatrix}1 & 1\\ -\sqrt{2} & \sqrt{2}\end{pmatrix}\begin{pmatrix}\alpha_1 \cos(\omega_1 t + \varphi_1)\\ \alpha_2 \cos(\omega_2 t + \varphi_2)\end{pmatrix}$$

Hence we might be tempted to try a coordinate transform

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}1 & 1\\ -\sqrt{2} & \sqrt{2}\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix} = \begin{pmatrix}b_1 + b_2\\ -\sqrt{2}\,b_1 + \sqrt{2}\,b_2\end{pmatrix}$$

in the ODEs:

$$2\ddot{\theta}_1 + \ddot{\theta}_2 = -\frac{2g}{\ell}\theta_1$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 = -\frac{g}{\ell}\theta_2$$

after some algebra:

$$\ddot{b}_2 = -\frac{g}{\ell}(2-\sqrt{2})\,b_2$$

$$\ddot{b}_1 = -\frac{g}{\ell}(2+\sqrt{2})\,b_1$$

meaning that the system is decoupled

We can see this also in $L$:

$$L = m\ell^2\dot\theta_1^2 + \frac{1}{2}m\ell^2\dot\theta_2^2 + m\ell\,\dot\theta_1\dot\theta_2 - mg\ell\,\theta_1^2 - \frac{1}{2}mg\ell\,\theta_2^2$$

$$\theta_1 = b_1 + b_2 \qquad \dot\theta_1 = \dot b_1 + \dot b_2$$

$$\theta_2 = -\sqrt{2}\,b_1 + \sqrt{2}\,b_2 \qquad \dot\theta_2 = -\sqrt{2}\,\dot b_1 + \sqrt{2}\,\dot b_2$$

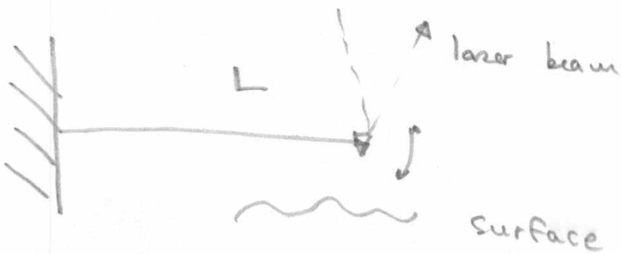
$$L = m\ell^2(\dot b_1 + \dot b_2)^2 + \frac{1}{2}m\ell^2(-\sqrt{2}\,\dot b_1 + \sqrt{2}\,\dot b_2)^2 + m\ell^2(\dot b_1 + \dot b_2)(-\sqrt{2}\,\dot b_1 + \sqrt{2}\,\dot b_2) - mg\ell\,(b_1 + b_2)^2 - \frac{1}{2}mg\ell\,(-\sqrt{2}\,b_1 + \sqrt{2}\,b_2)^2$$

$$= m\ell^2(\dot b_1^2 + \underline{2\dot b_1\dot b_2} + \dot b_2^2) + \frac{1}{2}m\ell^2(2\dot b_1^2 - \underline{4\dot b_1\dot b_2} + 2\dot b_2^2) + m\ell^2(-\sqrt{2}\,\dot b_1^2 + \underline{\sqrt{2}\,\dot b_1\dot b_2} - \underline{\sqrt{2}\,\dot b_1\dot b_2} + \sqrt{2}\,\dot b_2^2) - mg\ell\,(b_1^2 + \underline{2b_1b_2} + b_2^2) - \frac{1}{2}mg\ell\,(2b_1^2 - \underline{4b_1b_2} + 2b_2^2)$$

$$= m\ell^2(2-\sqrt{2})\,\dot b_1^2 + m\ell^2(2+\sqrt{2})\,\dot b_2^2 - 2mg\ell\,b_1^2 - 2mg\ell\,b_2^2$$

is clearly decoupled.

**Applied Problem:** Cantilever of Atomic-Force-Microscope




$$\varrho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^2} = 0$$

$x = 0$: clamped end: $y = 0$, $y_x = 0$

$x = L$: free end: $y_{xx} = 0$ $y_{xxx} = 0$

boundary conditions

General solution:

$$y(x,t) = (a_1 \sin kx + a_2 \cos kx + a_3 \sinh kx + a_4 \cosh kx)\, e^{-i\omega t}$$

Yields (a) dispersion relation: $-\omega^2 \varrho A + k^4 EI = 0$

(b) b.c. impose condition on $k$:

$$1 + \cos(kL) \cosh(kL) = 0$$

Solutions $k_n$ determine frequencies $\omega_n$