Analytical Dynamics

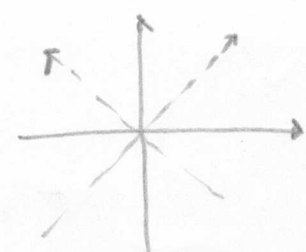
Lecture 5

Relationship between symmetries and conserved quantities:

__Example:__ L does not depend explicitly on t $\Rightarrow$ E is conserved

L does not depend on q_n $\Rightarrow$ $p_n = \frac{\partial L}{\partial \dot{q}_n}$ is conserved

$$L = \frac{1}{2} m (\dot{r}^2 + r\dot{\varphi}^2) - V(r)$$



we can rotate the coordinate system

__Remark:__ If L and L' differ by a __total__ derivative, the dynamics of the system are unchanged.

$$L' = L + \frac{d}{dt} f(q_i, t)$$

as $\delta S = 0 \Leftrightarrow \delta S' = 0$

__Proof:__

$$S = \int L(q_i, \dot{q}_i, t)\, dt$$

$$S' = \int L'(q_i, \dot{q}_i, t)\, dt$$

$$\delta S' - \delta S = \delta \int_{t_1}^{t_2} dt \frac{df}{dt} = \delta f(q_i, t) \Big|_{t_1}^{t_2}$$

$$= \sum_i \frac{\partial f}{\partial q_i} \delta q_i \Big|_{t_1}^{t_2} = 0$$

Point transformation:

$$\left.\begin{aligned} q_i' &= q_i + \epsilon \psi_i(q_j, \dot{q}_j, t) \\ t' &= t + \epsilon \phi(q_i, \dot{q}_i, t) \end{aligned}\right\} \quad (*)$$

$$L' = L(q_i', \dot{q}_i', t')$$

$$S' = \int_{t_1'}^{t_2'} dt' \, L(q_i', \dot{q}_i', t')$$

(At $\epsilon = 0$ we have $L(q_i', \dot{q}_i', t') = L(q_i, \dot{q}_i, t)$)

$$S' = \int_{t_1}^{t_2} dt \left(\frac{dt'}{dt} L(q_i', \dot{q}_i', t') \right)$$

$$\approx S + \epsilon \int_{t_1}^{t_2} dt \, \frac{d}{d\epsilon} \left(\frac{dt'}{dt} L(q_i', \dot{q}_i', t') \right) \Big|_{\epsilon=0}$$

The dynamics of the system will be unchanged if

$$\boxed{\frac{d}{d\epsilon} \left(\frac{dt'}{dt} L(q_i', \dot{q}_i', t') \right) \Big|_{\epsilon=0} = \frac{d}{dt} f(q_i, t)} \quad (**)$$

If such an f exists, we say that L is symmetric with respect to $(*)$.

Note: • If L does not depend on q_j, set

$q_j' = q_j + \epsilon$, hence $\psi_j = 1$, $\phi = 0$, $\psi_i = 0$ for $i \neq j$

as L does not change under

$q_j' = q_j + \epsilon$, $(**)$ is satisfied with $f = 0$.

(Formal calculation: $\frac{d}{d\epsilon}\left(\frac{dt'}{dt} L(q_i', \dot{q}_i', t)\right)$

$= \frac{d}{d\epsilon} L(q_1, \dots q_j + \epsilon, \dots, q_n, \dot{q}_i, t) = \frac{\partial L}{\partial q_j} = 0$

if L does not depend on q_j)

• If L does not depend on t, set

$t' = t + \epsilon$, hence $\psi_i = 0$, $\phi = 1$

$\Rightarrow$ $\frac{d}{d\epsilon}\left(\frac{dt'}{dt} L'\right) = \frac{\partial L}{\partial t} = 0$ if L does not depend explicitly on time.

If (**) is satisfied, we can construct explicitly a conserved quantity:

$$I = \sum_i \frac{\partial L}{\partial \dot{q}_i} \psi_i + \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \phi - f(q_i, t)$$

Note: For the above examples:

(a) $\frac{\partial L}{\partial q_j} = 0$ for a particular j: $\quad I = \frac{\partial L}{\partial \dot{q}_j} \equiv p_j$

(b) $\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad I = L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i$

as expected.

Proof of Noether's Theorem:

can depend on all q_j

$$t' = t + \epsilon \phi(q_i, \dot{q}_i, t) \qquad q_i' = q_i + \epsilon \psi_i (q_j, \dot{q}_j, t)$$

$$\frac{dt'}{dt} = 1 + \epsilon \frac{d\phi}{dt} \qquad \dot{q}_i' = \frac{dq_i'}{dt'} = \frac{dq_i'}{dt}\frac{dt}{dt'}$$

$$= \left(\frac{dq_i}{dt} + \epsilon \frac{d\psi_i}{dt} \right)\left(1 - \epsilon \frac{d\phi}{dt} \right)$$

$$= \dot{q}_i + \epsilon \left(\frac{d\psi_i}{dt} - \dot{q}_i \frac{d\phi}{dt} \right)$$

$$\frac{d}{d\epsilon}\left(\frac{dt'}{dt}L'\right)\Bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\left(\left(1+\epsilon\frac{d\phi}{dt}\right)L'\right)\Bigg|_{\epsilon=0}$$

$$= \frac{dL'}{d\epsilon}\Bigg|_{\epsilon=0} + L\frac{d\phi}{dt}$$

$$= \sum \frac{\partial L}{\partial q_i}\psi_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i'}{d\epsilon} + \frac{\partial L}{\partial t}\phi + L\frac{d\phi}{dt}$$

$$= \sum \frac{\partial L}{\partial q_i}\psi_i + \frac{\partial L}{\partial \dot{q}_i}\left(\frac{d\psi_i}{dt} - \dot{q}_i\frac{d\phi}{dt}\right) + \frac{\partial L}{\partial t}\phi + L\frac{d\phi}{dt}$$

$$= \sum \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\psi_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d\psi_i}{dt} + \left(L - \sum\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\frac{d\phi}{dt} + \frac{\partial L}{\partial t}\phi$$

$$= \frac{d}{dt}\left(\sum\frac{\partial L}{\partial \dot{q}_i}\psi_i\right) + \frac{d}{dt}\left[\left(L - \sum\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\phi\right]$$

(note: $\displaystyle\frac{d}{dt}\left(L - \sum\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) = \frac{\partial L}{\partial t}$)

$$\left[\ \frac{d}{dt}\left(L - \sum p_i\dot{q}_i\right) = \sum\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i - \dot{p}_i\dot{q}_i - p_i\ddot{q}_i + \frac{\partial L}{\partial t} = \frac{\partial L}{\partial t}\ \right]$$

Since $\frac{d}{dt} f(q_i, t) = \frac{d}{d\epsilon}\left(\frac{dt'}{dt} L'\right)\Big|_{\epsilon=0}$ we can integrate with respect to t and the theorem follows.

A more advanced example: $\boxed{\text{Laplace - Runge - Lenz Vector}}$

$$L = T - V = \frac{1}{2}\mu(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V(r)$$

with $V(r) = -\frac{\alpha}{r}$ is the kepler potential.

We will show that if we consider

$$x_i' = x_i + \frac{\epsilon}{2}(2p_i x_s - x_i p_s - \delta_{is}(\vec{r}\cdot\vec{p})) \qquad \vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \vec{p} = \mu\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$$

then δL is given by $\epsilon\mu\alpha \frac{d}{dt}\left(\frac{x_s}{r}\right)$

and we can use the Noether theorem to construct

$$A_s = (p^2 x_s - p_s(\vec{r}\cdot\vec{p})) - \mu\alpha\left(\frac{x_s}{r}\right)$$
$$= [\vec{p}\times(\vec{r}\times\vec{p})]_s - \mu\alpha\frac{x_s}{r}$$

For $s = 1, 2, 3$, the components of a conserved quantity (Laplace - Runge - Lenz vector)

Note: It is sufficient to work in a plane with $z = 0$ and to show this only for $s = 1$.

$$L = \frac{1}{2}\mu\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{\alpha}{\sqrt{x_1^2 + x_2^2}} \qquad \begin{aligned} x_1' &= x_1 - \frac{\epsilon}{2}\mu x_2 \dot{x}_2 \\ x_2' &= x_2 + \frac{\epsilon}{2}\mu\left(2\dot{x}_2 x_1 - x_2\dot{x}_1\right) \end{aligned}$$

$$\Rightarrow \dot{x}_1' = \dot{x}_1 - \frac{\epsilon}{2}\mu\left(\dot{x}_2^2 + x_2\ddot{x}_2\right)$$

$$\dot{x}_2' = \dot{x}_2 + \frac{\epsilon}{2}\mu\left(2\ddot{x}_2 x_1 + 2\dot{x}_2\dot{x}_1 - \dot{x}_2\dot{x}_1 - x_2\ddot{x}_1\right)$$

$$= \dot{x}_2 + \frac{\epsilon}{2}\mu\left(2\ddot{x}_2 x_1 + \dot{x}_1\dot{x}_2 - x_2\ddot{x}_1\right)$$

We use the Euler-Lagrange eqs to eliminate the $\ddot{\ }$-terms:

$$\mu\ddot{x}_1 = -\frac{\alpha x_1}{r^3} \qquad \mu\ddot{x}_2 = -\frac{\alpha x_2}{r^3} \qquad (*)$$

$$\dot{x}_1'^2 + \dot{x}_2'^2 - \left(\dot{x}_1^2 + \dot{x}_2^2\right) = -\epsilon\mu\dot{x}_1\left(\dot{x}_2^2 + x_2\ddot{x}_2\right) + \epsilon\mu\dot{x}_2\left(2\ddot{x}_2 x_1 + \dot{x}_1\dot{x}_2 - x_2\ddot{x}_1\right)$$

$$= -\epsilon\mu x_2\dot{x}_1\ddot{x}_2 + 2\epsilon\mu x_1\dot{x}_2\ddot{x}_2 - \epsilon\mu x_2\dot{x}_2\ddot{x}_1$$

$$\overset{(*)}{=} \epsilon\alpha\frac{x_2^2\dot{x}_1}{r^3} - 2\alpha\epsilon\frac{x_1\dot{x}_2 x_2}{r^3} + \alpha\epsilon\frac{x_2\dot{x}_2 x_1}{r^3} = \epsilon\alpha\left(\frac{x_2^2\dot{x}_1}{r^3} - \frac{x_1 x_2\dot{x}_2}{r^3}\right)$$

As $\delta L = \delta T - \delta V$, we have found

$$\delta T = \frac{1}{2}\mu\epsilon\alpha\left(\frac{x_2^2\dot{x}_1}{r^3} - \frac{x_1 x_2\dot{x}_2}{r^3}\right)$$

To compute δV, we see

$$\frac{\alpha}{r'} = \frac{\alpha}{r} - \frac{\alpha}{r^2}\delta r \,, \quad \delta r = \frac{x_1 \delta x_1}{r} + \frac{x_2 \delta x_2}{r} \quad \text{as} \quad r = \sqrt{x_1^2 + x_2^2}$$

$$\Rightarrow \delta V = +\frac{\alpha}{r^3}\left(x_1\left(-\frac{\epsilon}{2}\mu x_2 \dot{x}_2\right) + x_2\left(\frac{\epsilon}{2}\mu\left(2\dot{x}_2 x_1 - x_2 \dot{x}_1\right)\right)\right)$$

$$= -\frac{\mu\alpha\epsilon}{2r^3}\left[x_1 x_2 \dot{x}_2 - 2\dot{x}_2 x_1 x_2 + x_2^2 \dot{x}_1\right] = \frac{-\mu\alpha\epsilon}{2r^3}\left[x_2^2 \dot{x}_1 - \dot{x}_2 x_1 x_2\right]$$

$$\Rightarrow \delta L = \delta T - \delta V = \frac{\mu\alpha\epsilon}{2r^3}\left[2x_2^2 \dot{x}_1 + x_1 x_2 \dot{x}_2\right]$$

$$= \frac{\mu\alpha\epsilon}{r^3}\left[\dot{x}_1 x_2^2 + x_1 x_2 \dot{x}_2\right]$$

On the other hand: $\frac{d}{dt}\left(\frac{x_1}{r}\right) = \frac{\dot{x}_1 r - x_1 \dot{r}}{r^2} =$

$$\frac{\dot{x}_1 r^2}{r^3} - x_1 \frac{x_1 \dot{x}_1 + x_2 \dot{x}_2}{r^3} = \frac{\dot{x}_1 (x_1^2 + x_2^2) - x_1^2 \dot{x}_1 - x_1 x_2 \dot{x}_2}{r^3}$$

$$= \frac{\dot{x}_1 x_2^2 - x_1 x_2 \dot{x}_2}{r^3} \quad \Rightarrow \quad \delta L = \mu\alpha\epsilon \frac{d}{dt}\left(\frac{x_1}{r}\right)$$

From the Noether-theorem we find that

$$A_1 = \sum \frac{\partial L}{\partial \dot{x}_k} \psi_k - \mu\alpha \frac{x_1}{r} \quad \text{is conserved.}$$

Here: $\psi_1 = -\frac{1}{2}\mu x_2 \dot{x}_2 \quad \psi_2 = \frac{1}{2}\mu(2\dot{x}_2 x_1 - x_2\dot{x}_1)$

$$\frac{\partial L}{\partial \dot{x}_1} = \mu \dot{x}_1 \quad \frac{\partial L}{\partial \dot{x}_2} = \mu \dot{x}_2$$

$$\Rightarrow A_1 = -\frac{1}{2}\mu^2 x_2 \dot{x}_2 \dot{x}_1 + \frac{1}{2}\mu^2(2\dot{x}_2^2 x_1 - \dot{x}_1\dot{x}_2 x_2) - \mu\alpha\frac{x_1}{r}$$

$$= -\mu^2 x_2\dot{x}_2\dot{x}_1 + \mu^2 x_1 \dot{x}_2^2 - \mu\alpha\frac{x_1}{r}$$

On the other hand:

$$p^2 x_1 - p_1(\vec{r}\cdot\vec{p}) = \mu^2(\dot{x}_1^2 + \dot{x}_2^2)x_1 - \mu^2\dot{x}_1(x_1\dot{x}_1 + x_2\dot{x}_2)$$

$= \mu^2 x_1 \dot{x}_2^2 - \mu^2 \dot{x}_1\dot{x}_2 x_2$ which completes the proof of the statement.