Two-Body Problem

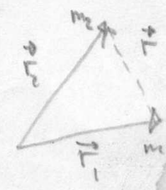
Assume $V = V(|\vec{r}|) = V(r)$ $\qquad L = \frac{m_1}{2}\dot{\vec{r}}_1^{\,2} + \frac{m_2}{2}\dot{\vec{r}}_2^{\,2} - V(r)$

We start with 6 coordinates, given by $\vec{r}_1$ and $\vec{r}_2$.

First, we show how the problem can be reduced to an equivalent one-body problem by using the center of mass.

$$(\vec{r}_1, \vec{r}_2) \longrightarrow (\vec{R}, \vec{r}) \quad \text{where} \quad \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \qquad \vec{r} = \vec{r}_2 - \vec{r}_1 \quad (*)$$

Then $T = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^{\,2}$, $\quad \mu = \frac{m_1 m_2}{m_1+m_2}$



(called "reduced mass")

This is a quick calculation:

First, (*) yields $(m_1+m_2)\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$, $\quad m_1\vec{r} + m_1\vec{r}_1 = m_1\vec{r}_2$

$$m_1\vec{r}_1 = m_1\vec{r}_2 - m_1\vec{r}$$

$$(m_1+m_2)\vec{R} = (m_1+m_2)\vec{r}_2 - m_1\vec{r}$$

$$\vec{R} = \vec{r}_2 - \frac{m_1}{m_1+m_2}\vec{r} \Rightarrow \vec{r}_2 = \vec{R} + \frac{m_1}{m_1+m_2}\vec{r}$$

similar way: $\vec{r}_1 = \vec{R} - \frac{m_2}{m_1+m_2}\vec{r}$

$$T = \frac{1}{2}m_1\left(\dot{\vec{R}} - \frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} + \frac{m_1}{m_1+m_2}\dot{\vec{r}}\right)^2 = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{r}^2$$

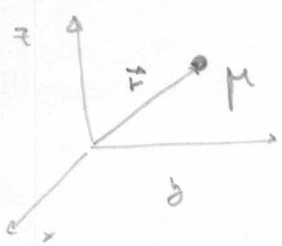
$$\left[\text{as } \frac{m_1 m_2^2}{(m_1+m_2)^2} + \frac{m_2 m_1^2}{(m_1+m_2)^2} = \frac{m_1 m_2}{m_1+m_2} = \mu \right]$$

$$\Rightarrow \quad L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu \dot{\vec{r}}^2 - V(r)$$

Now $\frac{\partial L}{\partial R_k} = 0 \Rightarrow \vec{R}$ is cyclic, $\vec{P} = \frac{\partial L}{\partial \dot{\vec{R}}} = \text{const.} = M\dot{\vec{R}}$

$\vec{R} = \vec{R}_0 + \vec{V}_0 t$ and we can switch to a coordinate system in which $\vec{R}_0, \vec{V}_0$ are zero. From here on:

$L = \frac{1}{2}\mu \dot{\vec{r}}^2 - V(r)$ an equivalent one-body problem.



Central force $\Rightarrow \vec{L} = \vec{r} \times \vec{p}$ is conserved

So, $\vec{r}$ always $\perp \vec{L}$, but $\vec{L}$ is fixed

$\Rightarrow$ motion must take place in a plane

Now we choose the coordinate system such that (x,y) is that plane

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r)$$

Now φ is cyclic (see above)

Aim: Explain Kepler's Laws (1606) : (1) and (2)
(1619) : (3)

(1) All planets move in elliptical orbits

(2) A line that connects a planet to the sun sweeps out equal areas in equal times

(3) The square of the period of any planet is proportional to the cube of the semi major axis of its orbit.

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r)$$, φ is cyclic $\Rightarrow$ $p_\varphi \equiv \ell = \text{const.}$

$$\ell = \frac{\partial L}{\partial \dot{\varphi}} = \mu r^2 \dot{\varphi} \Rightarrow \dot{\varphi} = \frac{\ell}{\mu r^2}$$

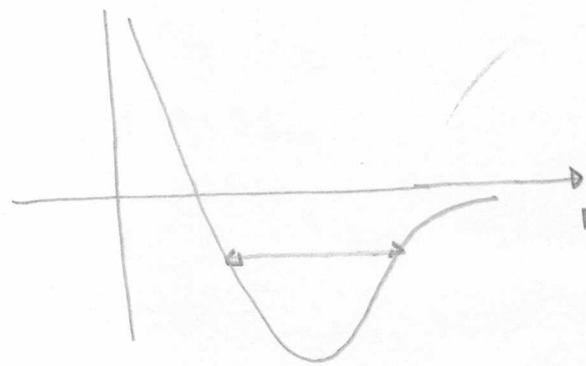
eq for r: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \Rightarrow \mu\ddot{r} - \mu r\dot{\varphi}^2 + V'(r) = 0$$

$$\mu\ddot{r} - \frac{\ell^2}{\mu r^3} + V'(r) = \mu\ddot{r} + \frac{d}{dr}\left(\frac{\ell^2}{2\mu r^2} + V'\right) = 0$$

$$\mu\ddot{r} = -\frac{d}{dr}V_{eff}(r) \qquad V_{eff} = \frac{\ell^2}{2\mu r^2} + V(r)$$ effective potential

Kepler problem: $\frac{1}{r^2}$ force $\Rightarrow V = -\frac{\alpha}{r}$, $\alpha > 0$

$$\Rightarrow V_{eff}(r) = \frac{\ell^2}{2\mu r^2} - \frac{\alpha}{r}$$



bounded motion for $E < 0$

$$\frac{1}{2}\mu \dot{r}^2 + V_{eff} = E \Rightarrow \dot{r} = \sqrt{\frac{2}{\mu}(E - V_{eff})}$$

$$t - t_0 = \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{\mu}(E - V_{eff}(r'))}}$$ is not easy to solve.

Trick: we use $r = r(\varphi)$. $\dot{r} = \frac{dr}{d\varphi}\dot{\varphi} \equiv r'\dot{\varphi}$, $\dot{\varphi} = \frac{\ell}{\mu r^2}$

Remember: $$\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{\alpha}{r} = E$$

$$\frac{1}{2}\mu(r'^2 + r^2)\dot{\varphi}^2 - \frac{\alpha}{r} = E$$

$$\frac{1}{2}\mu(r'^2 + r^2)\frac{\ell^2}{\mu^2 r^4} - \frac{\alpha}{r} = E$$

$$\frac{(r'^2 + r^2)}{r^4} = \left(E + \frac{\alpha}{r}\right)\frac{2\mu}{\ell^2}$$

Set $u = \frac{1}{r}$ $\quad r' = -\frac{1}{u^2} u' = -r^2 u' \Rightarrow$

$$u'^2 + u^2 = \frac{2\mu}{\ell^2}(E + \alpha u)$$

$$u'^2 + u^2 - \frac{2\alpha\mu}{\ell^2} u = \frac{2\mu E}{\ell^2}$$

$$u'^2 + \left(u - \frac{\alpha\mu}{\ell^2}\right)^2 = \frac{2\mu E}{\ell^2} + \frac{\alpha^2\mu^2}{\ell^4} \quad \Big| \cdot \frac{\ell^4}{\alpha^2\mu^2} \equiv \varkappa^2$$

$$(\varkappa u')^2 + (\varkappa u - 1)^2 = \frac{2E\ell^2}{\alpha^2\mu} + 1 \quad , \text{ set } v = \varkappa u - 1$$

$$v'^2 + v^2 = \frac{2E\ell^2}{\alpha^2\mu} + 1 = \epsilon^2$$

$\Rightarrow$ $v = \epsilon \cos(\varphi - \varphi_0) = \varkappa u - 1$ or $u = \frac{1 + \epsilon\cos(\varphi - \varphi_0)}{\varkappa}$

$\Rightarrow$ $$\boxed{r = \frac{\varkappa}{1 + \epsilon\cos(\varphi - \varphi_0)}}$$

Conic section:

$\epsilon > 1$: hyperbola

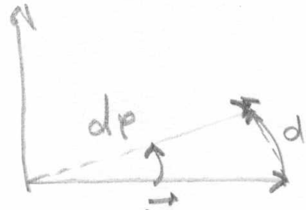
$\epsilon = 1$: parabola

$\epsilon < 1$: ellipse

(eccentricity)

$\Rightarrow$ 1st Kepler Law

For the 2^{nd} Kepler Law, we note that the area dS that is swept out in dt is



$dr = r\, d\varphi$

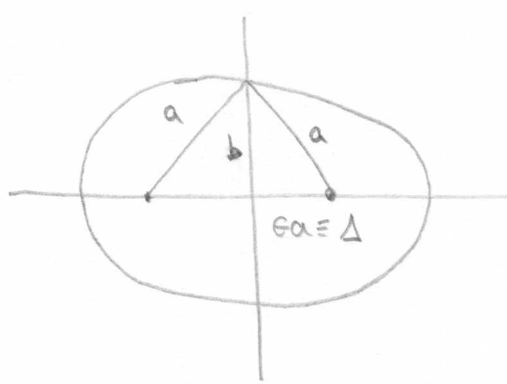
$$dS = \frac{1}{2}|\vec{r} \times d\vec{r}|$$

$$= \frac{1}{2} r^2 d\varphi = \frac{1}{2} r^2 \dot{\varphi}\, dt$$

$$\Rightarrow \quad S = \int_{t_0}^{t_1} dS = \frac{1}{2}\int_{t_0}^{t_1} r^2 \dot{\varphi}\, dt \qquad \text{but } r^2\dot{\varphi} = \frac{\ell}{\mu}$$

$$= \frac{1}{2}\frac{\ell}{\mu}(t_1 - t_0)$$ which proves 2^{nd} Law.

The third law needs a little more work



$$2a = r_1 + r_2 \qquad \epsilon^2 = 1 + \frac{2E\ell^2}{\alpha^2\mu}$$

after proper choice of φ_0:

$$r_{min} = r(0) = \frac{\varkappa}{1+\epsilon}, \quad r_{max} = r(\pi) = \frac{\varkappa}{1-\epsilon}$$

$$\Rightarrow \quad a = \frac{1}{2}(r_{min} + r_{max}) = \frac{\varkappa}{2}\left(\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right) = \frac{\varkappa}{1-\epsilon^2}$$

$$\Delta = \frac{1}{2}(r_{max} - r_{min}) = \frac{\varkappa}{2}\left(\frac{1}{1-\epsilon} - \frac{1}{1+\epsilon}\right) = \frac{\epsilon\varkappa}{1-\epsilon^2} = \epsilon a$$

$$b^2 + \Delta^2 = a^2 \Rightarrow b = a\sqrt{1-\epsilon^2}$$

Now we can compute $S = \pi a b = \frac{\ell}{2\mu} T$ and find out T. Remember $\varkappa = \frac{\ell^2}{\alpha\mu}$, $\epsilon^2 = 1 + \frac{2E\ell^2}{\alpha^2\mu}$, $E<0$

$$a = \frac{\varkappa}{1-\epsilon^2} = \frac{\ell^2}{\alpha\mu}\,\frac{\alpha^2\mu}{2|E|\ell^2} = \frac{\alpha}{2|E|} \Rightarrow |E| = \frac{\alpha}{2a}$$

$$b = \sqrt{1-\epsilon^2}\, a = \sqrt{\frac{2|E|\ell^2}{\alpha^2\mu}} \cdot \frac{\alpha}{2|E|} = \frac{\ell}{\sqrt{2\mu|E|}}$$

As $S = \pi a b = \frac{\ell}{2\mu} T \Rightarrow T = \frac{2\mu a b \pi}{\ell}$

$$= \frac{\cancel{2}\mu\pi}{\cancel{\ell}} \cdot \frac{\alpha}{\cancel{2}|E|} \cdot \frac{\cancel{\ell}}{\sqrt{2\mu|E|}} = \frac{\alpha\mu}{\sqrt{2\mu}} \cdot \frac{\pi}{|E|^{3/2}} = \frac{\alpha\mu}{\sqrt{2\mu}}\,\frac{\pi}{\alpha^{3/2}} \cdot 2^{3/2} \cdot a^{3/2}$$

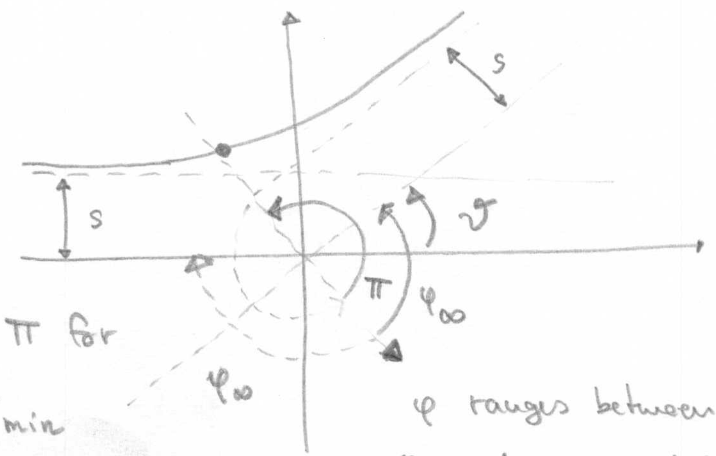
$$= 2\pi\sqrt{\frac{\mu}{\alpha}}\, a^{3/2}$$

now $\mu \approx m_{planet}$, $\alpha \approx G\, m_{planet}\, m_{\odot}$

$\Rightarrow$ 3rd Law $\quad (T^2/a^3 = \text{const})$

Scattering: $\epsilon > 1$: hyperbola $\qquad r = \dfrac{\varkappa}{1 + \epsilon \cos\varphi}$

Very important: "convenient" choice of coordinate system



$\varphi = \pi$ for $r = r_{min}$

φ ranges between φ_∞ and π, π and $2\pi - \varphi_\infty$

$$\vec{r} = x\vec{e}_x + s\vec{e}_y$$
$$\vec{v} = v_0 \vec{e}_x$$
$$\vec{L} = \mu \vec{r} \times \vec{v} = -\mu s v_0 \vec{e}_z$$
$$\ell = -\mu s v_0, \quad E = \frac{1}{2}\mu v_0^2$$
$$\epsilon^2 = \frac{2E\ell^2}{\alpha^2\mu} + 1$$
$$\varkappa = \frac{\ell^2}{\alpha\mu}$$

s is called impact parameter

We can express $\varkappa$ and ϵ in terms of E and s:

$$\varkappa = \frac{\ell^2}{\alpha\mu} = \frac{s^2 v_0^2 \mu^2}{\alpha\mu} = \frac{2Es^2}{\alpha} \quad \text{and} \quad \epsilon^2 = \frac{2Es^2v_0^2\mu^{\cancel{2}}}{\alpha^2\cancel{\mu}} + 1$$
$$= 1 + \left(\frac{2Es}{\alpha}\right)^2$$

repulsive case: $\alpha < 0 \Rightarrow \varkappa < 0$

As $r \geq 0$, we need $1 + \epsilon\cos\varphi \leq 0$

$$\vartheta = 2\varphi_\infty - \pi \quad \Rightarrow \quad \varphi_\infty = \frac{\vartheta}{2} + \frac{\pi}{2}$$

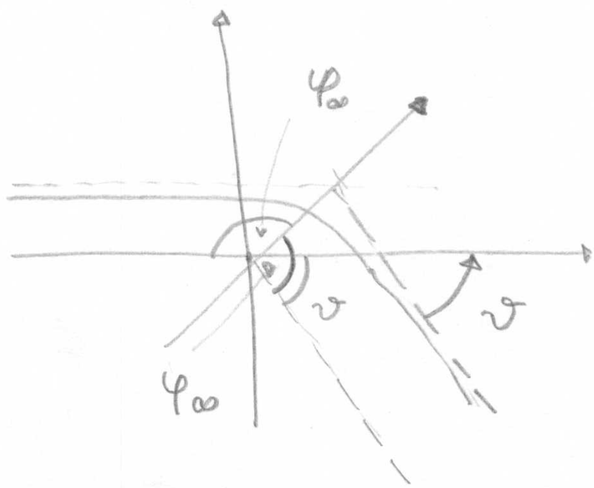
$r \to \infty$: $0 = 1 + \epsilon\cos\varphi_\infty$
$= 1 + \epsilon\cos\left(\frac{\vartheta}{2} + \frac{\pi}{2}\right)$

$$\Rightarrow 1 - \epsilon\sin\frac{\vartheta}{2} = 0 \quad \text{or} \quad \boxed{\sin\frac{\vartheta}{2} = \frac{1}{\epsilon}}$$

attractive case:

$\alpha > 0$, $\varkappa > 0$

Choose coordinate system such that $\varphi = 0$ for $r = r_{min}$



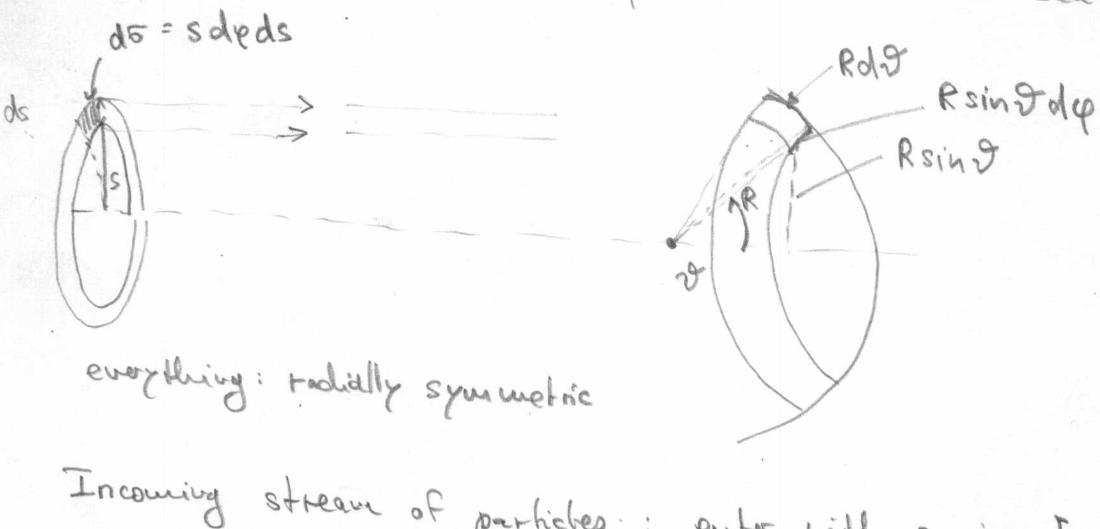
again $\pi + \vartheta = 2\varphi_\infty$

range: $-\varphi_\infty \leq \varphi \leq \varphi_\infty$

as $\alpha > 0$, $\varkappa > 0$: $1 + \epsilon \cos\varphi \geq 0$

Rutherford Scattering:

Consider incoming stream of particles that are scattered.



$dA = R^2 \sin\vartheta\, d\vartheta\, d\varphi$

It depends on ϑ how much impact $d\varphi$ has

everything: radially symmetric

Incoming stream of particles: enter with s in $[s, s+ds]$

Outgoing stream: scattering in $[\vartheta, \vartheta + d\vartheta]$

We assume that no particles are lost:

Number of particles that go through $d\sigma$: $dI = j\,d\sigma$

$j = \frac{dI}{d\sigma}$ is the stream of particles (flux)

$d\sigma = s\, d\varphi\, ds \qquad j = \text{const.}$

detector: $d\Theta = R\,d\vartheta\, R\sin\vartheta\, d\varphi = R^2 d\Omega$

(element of solid angle) $\qquad d\Omega = \sin\vartheta\, d\vartheta\, d\varphi$

$$\Rightarrow \frac{dI}{d\Omega} = j\frac{d\sigma}{d\Omega}, \qquad \frac{1}{j}\frac{dI}{d\Omega} = \underbrace{\frac{d\sigma}{d\Omega}}_{\text{differential cross section}} = \left|\frac{s\,ds\,d\varphi}{\sin\vartheta\, d\vartheta\, d\varphi}\right|$$

$$\frac{d\sigma}{d\Omega} = \frac{s}{|\sin\vartheta|}\left|\frac{ds}{d\vartheta}\right|$$

$\Rightarrow$ we need $s = s(\vartheta)$ in order to be able to compare to experiments.

$$\epsilon^2 = \frac{1}{\sin^2\left(\frac{\vartheta}{2}\right)}, \qquad \left(\frac{2Es}{\alpha}\right)^2 = \epsilon^2 - 1 = \frac{1-\sin^2\frac{\vartheta}{2}}{\sin^2\frac{\vartheta}{2}} = \left(\frac{\cos\vartheta/2}{\sin\frac{\vartheta}{2}}\right)^2$$

$$s = \frac{|\alpha|}{2E}\frac{\cos\vartheta/2}{\sin\vartheta/2} \qquad \frac{ds}{d\vartheta} = -\frac{|\alpha|}{2E\cdot 2}\frac{1}{\sin^2\vartheta/2}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \underbrace{\frac{|\alpha|}{4E}\frac{1}{\sin^2\frac{\vartheta}{2}}}_{\left|\frac{ds}{d\vartheta}\right|} \cdot \underbrace{\frac{|\alpha|}{2E}\frac{\cos\vartheta/2}{\sin\vartheta/2}}_{s} \underbrace{\frac{1}{|\sin\vartheta|}}_{=\frac{1}{2\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2}}} \sim \frac{1}{\sin^4\frac{\vartheta}{2}}$$

Remark: $\sigma = \int d\Omega \frac{d\sigma}{d\Omega} \sim \int_0^{2\pi} \int_0^{\pi} d\vartheta \sin\vartheta \frac{1}{\sin^4 \frac{\vartheta}{2}} d\varphi \to \infty$

diverges. If one accounts for scattering by electrons: Finite value

Comparison: Scattering on a solid sphere

$\pi - 2\varphi = \vartheta$

$$s = R \sin\varphi = R \sin\left(\frac{\pi}{2} - \frac{\vartheta}{2}\right) = R \cos\frac{\vartheta}{2}$$

$$ds = -\frac{R}{2} \sin\frac{\vartheta}{2} d\vartheta, \quad \frac{d\sigma}{d\Omega} = \frac{s}{\sin\vartheta} \cdot \left| -\frac{R}{2} \sin\frac{\vartheta}{2} \right|$$

$$= \frac{R \cos\frac{\vartheta}{2}}{\sin\vartheta} \frac{R}{2} \sin\frac{\vartheta}{2} = \frac{R^2}{4}$$

total cross-section: $\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = 4\pi \frac{R^2}{4} = \pi R^2$

(as expected)

Remark: Inverse scattering transform