Analytical Dynamics

We can also derive Lagrange's equation using a different approach. First we need to review the calculus of variations:

$$I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx \quad \text{is a } \underline{\text{Functional}}$$

(maps a function $y = y(x)$ to a real number)

Typical question: Find y for which $I(y)$ takes a maximum or minimum,

Idea: consider a small variation $y_\epsilon(x) = y^{(0)}(x) + \epsilon h(x)$

$$I(y_\epsilon) = \int_{x_1}^{x_2} f(x, y^{(0)} + \epsilon h(x), y^{(0)\prime} + \epsilon h)\, dx$$

$$\delta I = I(y_\epsilon) - I(y^{(0)}) = \epsilon \int_{x_1}^{x_2} \frac{\partial f}{\partial y} h + \frac{\partial f}{\partial y'} h'\, dx + \mathcal{O}(\epsilon^2)$$

we assume that endpoints are fixed: $h(x_1) = h(x_2) = 0$

$\delta I \Rightarrow$ we can integrate by parts and find

$$I(y_\epsilon) - I(y^{(0)}) = \epsilon \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) h \, dx + \underbrace{\frac{\partial f}{\partial y'} h \Big|_{x_1}^{x_2}}_{=0}$$

$\Rightarrow$ If y is an extremum of I $\Rightarrow$ $\delta I(y) = 0$

or $$\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

Derivation of Lagrange equations:

The motion of the

Define $$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt \quad \text{(action integral)}$$

$$\delta S = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Compute $$\delta L = L(q_i + \epsilon v_i, \dot{q}_i + \epsilon \dot{v}_i, t) - L(q_i, \dot{q}_i, t)$$

$$= \epsilon \sum_i \frac{\partial L}{\partial q_i} v_i + \frac{\partial L}{\partial \dot{q}_i} \dot{v}_i$$

$$\Rightarrow \delta S = \epsilon \int_{t_1}^{t_2} \left(\sum_i \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) v_i \, dt = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Note: $\dfrac{\partial L}{\partial q_k} = 0 \Rightarrow p_k = \dfrac{\partial L}{\partial \dot{q}_k}$ is conserved

Hamilton's Principle: The motion of the system from time t_1 to t_2 is such that the action integral

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt \quad , \quad L = T - V$$

has a stationary value for the actual path of the motion.

Velocity dependent potentials: Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ is velocity dependent. Still we can use a Lagrangian formulation:

$$L = \frac{1}{2} m v^2 - q(\phi - \vec{v} \cdot \vec{A}) = T - \mathcal{U} \qquad (*)$$

where $E = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$, $B = \nabla \times \vec{A}$ are scalar and vector potential

$$\left(\nabla \times \vec{A} = \begin{pmatrix} A_{3y} - A_{2z} \\ A_{1z} - A_{3x} \\ A_{2x} - A_{1y} \end{pmatrix} \right)$$

Idea: $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \leftarrow$ generalized force

Set $\mathcal{U} = q\phi - q \vec{v} \cdot \vec{A}$ and show that $\frac{d}{dt} \frac{\partial \mathcal{U}}{\partial \dot{x}} - \frac{\partial \mathcal{U}}{\partial x}$ is the x-component of the Lorentz force

Let's compute $\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} - \frac{\partial U}{\partial x}$, $U = q\phi - q(\dot{x}A_1 + \dot{y}A_2 + \dot{z}A_3)$

$$= -q\frac{d}{dt}A_1 - q\frac{\partial \phi}{\partial x} + q(\dot{x}A_{1x} + \dot{y}A_{2x} + \dot{z}A_{3x})$$

$$= -q(\cancel{\dot{x}A_{1x}} + \dot{y}A_{1y} + \dot{z}A_{1z}) - q\frac{\partial \phi}{\partial x} + q(\cancel{\dot{x}A_{1x}} + \dot{y}A_{2x} + \dot{z}A_{3x}) - q\frac{\partial A_1}{\partial t}$$

$$= q\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_1}{\partial t}\right] + q\left[\dot{y}(A_{2x} - A_{1y}) + \dot{z}(A_{3x} - A_{1z})\right]$$

$$= qE_1 + q\left[\dot{y}B_3 - \dot{z}B_2\right] = (q\vec{E} + q\vec{v}\times\vec{B})_1$$

Particle in r-dependent potential

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r)$$

$x = r\cos\varphi$ $\dot{x} = \dot{r}\cos\varphi - r\sin\varphi\dot{\varphi}$

$y = r\sin\varphi$ $\dot{y} = \dot{r}\sin\varphi + r\cos\varphi\dot{\varphi}$

$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2)$

(*) $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$

(**) $\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$ but $\frac{\partial L}{\partial \varphi} = 0$ (φ is cyclic)

$\Rightarrow$ $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} = \ell$ is conserved (angular momentum)

From (*) we get $\frac{d}{dt}(m\dot{r}) - mr\dot{\varphi}^2 + \frac{dV}{dr} = 0$

$m\ddot{r} = mr\dot{\varphi}^2 - \frac{dV}{dr}$ and $\dot{\varphi} = \frac{\ell}{mr^2}$

$$m\ddot{r} = mr\frac{\ell^2}{m^2r^4} - \frac{dV}{dr} = \frac{\ell^2}{mr^3} - \frac{dV}{dr} = -\frac{d}{dr}\left(V + \frac{1}{2}\frac{\ell^2}{mr^2}\right) = -\frac{dV_{eff}}{dr}$$

Energy conservation for t-independent L

Assume that L does not depend explicitly on t.

$$L = T - V \qquad V = V(q_1, \dots, q_s) \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Consider $H = \sum p_i \dot{q}_i - L(q_1, \dots, q_s; \dot{q}_1, \dots, \dot{q}_s)$

$$\frac{dH}{dt} = \sum_i \dot{p}_i \dot{q}_i - p_i \ddot{q}_i - \frac{\partial L}{\partial q_i}\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = 0$$

as $\dot{p}_i = \frac{\partial L}{\partial q_i}$ from $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

Shortest distance between two points:

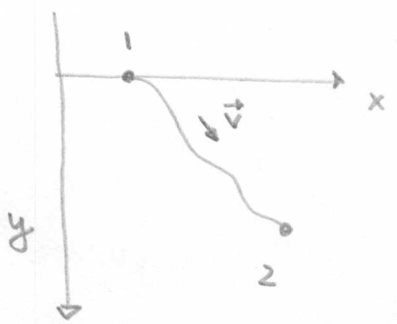
$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \dot{y}^2}\, dx$$

$\Rightarrow$ $I = \int_{x_1}^{x_2} \sqrt{1 + \dot{y}^2}\, dx$ shall be minimized $L = \sqrt{1 + \dot{y}^2}$

y is cyclic $\Rightarrow$ $\frac{\partial L}{\partial \dot{y}} = c = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$ $\Rightarrow$ $\dot{y} = a$, $y = ax + b$

* Brachistochrone problem

Find the curve for which a particle falling from rest under influence of gravity travels from the higher to the lower point in the least time.



$$t_{12} = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{\sqrt{1+y'^2}\,dx}{v}$$

Energy conservation: (at release $E=0$)

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

$$\Rightarrow t_{12} = \int_1^2 \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx = \frac{1}{\sqrt{2g}} \int_1^2 \sqrt{\frac{1+y'^2}{y}}\,dx$$

We need to consider $L = \sqrt{\dfrac{1+y'^2}{y}}$. To integrate, use that $H = y'\dfrac{\partial L}{\partial y'} - L$ is conserved as $\dfrac{\partial L}{\partial x} = 0$.

$$y' \frac{y'}{\sqrt{y(1+y'^2)}} - \sqrt{\frac{1+y'^2}{y}} = \text{const} = \frac{1}{A}$$

$$\frac{-1}{\sqrt{y(1+y'^2)}} = \frac{1}{A} \quad \text{or} \quad \frac{1}{y(1+y'^2)} = \frac{1}{A^2}$$

$$\frac{A^2}{y} = 1 + y'^2 \Rightarrow \frac{A^2}{y} - 1 = y'^2 \Rightarrow \frac{dy}{\sqrt{\frac{A^2}{y} - 1}} = dx$$

$$\sqrt{\frac{y}{A^2 - y}}\, dy = dx \qquad y = \frac{A^2}{2}(1 - \cos u) = A^2 \sin^2 \frac{u}{2}$$

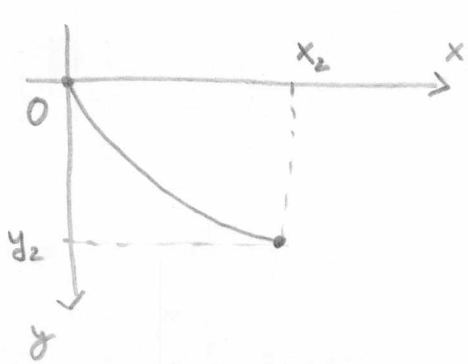
$$dy = A^2 \sin\frac{u}{2} \cos\frac{u}{2}\, du = \frac{A^2}{2} \sin u\, du$$

$$\int \sqrt{\frac{y}{A^2 - y^2}}\, dy = \int \sqrt{\frac{\frac{A^2}{2}(1-\cos u)}{A^2 - \frac{A^2}{2}(1-\cos u)}}\ \frac{A^2}{2} \sin u\, du$$

$$= \int \sqrt{\frac{1 - \cos u}{1 + \cos u}}\ \frac{A^2}{2} \sin u\, du = \int \frac{\sin u/2}{\cancel{\cos u/2}} A^2 \sin\frac{u}{2} \cancel{\cos\frac{u}{2}}\, du$$

$$= A^2 \int \sin^2 \frac{u}{2}\, du = \frac{A^2}{2} \int 1 - \cos u\, du = \frac{A^2}{2}(u - \sin u)$$

$$\Rightarrow \quad x = a(u - \sin u), \quad y = a(1 - \cos u), \quad a = \frac{A^2}{2}$$



Cycloid solution to brachistochrone problem