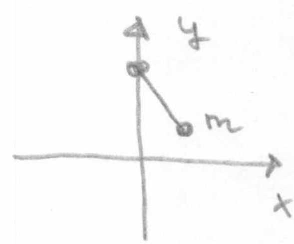
The Principle of d'Alembert

We can find equations of motion using Newton's Law.
Complexity depends on # of dimensions.
Often: motion is constrained

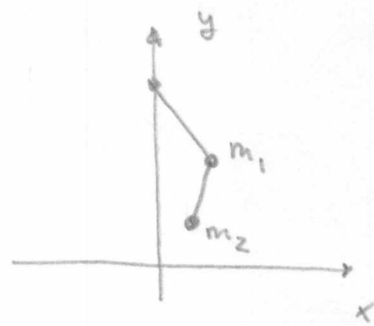
Ex: pendulum



$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$

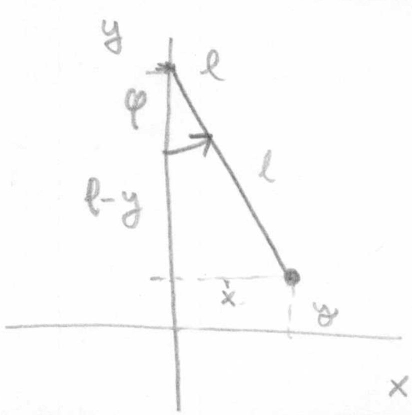
two coordinates
one constraint
= one degree of freedom

double pendulum



4 coordinates
2 constraints
= two degrees of freedom

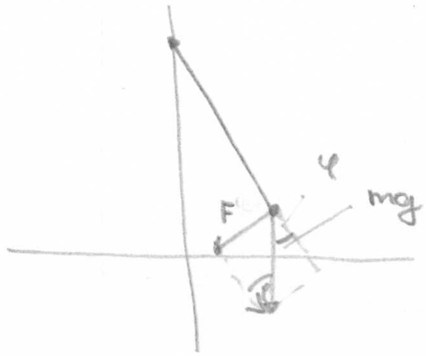
Idea: Find coordinates that can be varied independently (here: angles)



$x^2 + (y-\ell)^2 = \ell^2$ constraint

$x = \ell \sin\varphi$

$\ell - y = \ell \cos\varphi \Rightarrow y = \ell - \ell\cos\varphi$



$$F = -mg\sin\varphi$$

$$m l \ddot{\varphi} = -mg\sin\varphi$$

$$\boxed{\ddot{\varphi} + \frac{g}{\ell}\sin\varphi = 0}$$

(note: small φ : $\sin\varphi \approx \varphi \rightarrow$ harmonic oscillator, $\omega_0^2 = \frac{g}{\ell}$)

How can we generalize this idea?

Principle of d'Alembert:

A virtual displacement $\delta\vec{r}_i$ is a displacement that is compatible with the constraining forces.

The forces cannot produce virtual work

$$\sum_i \vec{F}_i^{(c)} \cdot \delta\vec{r}_i = 0$$

Now $\vec{F}_i^{(c)} + \vec{F}_i^{(e)} = \vec{F}_i = \dot{\vec{p}}_i$ or $\vec{F}_i^{(c)} = \dot{\vec{p}}_i - \vec{F}_i^{(e)}$

$$\Rightarrow \quad \boxed{\sum_i \left(\dot{\vec{p}}_i - \vec{F}_i^{(e)}\right) \cdot \delta\vec{r}_i = 0}$$

We can actually use this to derive eqs. of motion:

Again: Pendulum

$$\vec{F}^{(e)} = \begin{pmatrix} 0 \\ -mg \end{pmatrix} \qquad \delta\vec{r} = \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

$$\left(\vec{F}^{(e)} - \ddot{\vec{p}}\right) \cdot \delta\vec{r} = 0$$

$$-mg\,\delta y - m\ddot{x}\,\delta x - m\ddot{y}\,\delta y = 0$$

$$x = l\sin\varphi \qquad y = l - l\cos\varphi$$

$$\delta x = l\cos\varphi\, d\varphi \qquad \delta y = l\sin\varphi\, d\varphi$$

$$\dot{x} = l\dot{\varphi}\cos\varphi \qquad \ddot{x} = -l\dot{\varphi}^2\sin\varphi + l\ddot{\varphi}\cos\varphi$$

$$\dot{y} = l\dot{\varphi}\sin\varphi \qquad \ddot{y} = l\dot{\varphi}^2\cos\varphi + l\ddot{\varphi}\sin\varphi$$

$$\Rightarrow -mg\,l\sin\varphi\, d\varphi = m\left(-l\dot{\varphi}^2\sin\varphi + l\ddot{\varphi}\cos\varphi\right)\left(l\cos\varphi\, d\varphi\right)$$
$$+ m\left(l\dot{\varphi}^2\cos\varphi + l\ddot{\varphi}\sin\varphi\right)\left(l\sin\varphi\, d\varphi\right)$$

$$-mgl\sin\varphi\, d\varphi = ml^2\ddot{\varphi}\, d\varphi$$

$$\Rightarrow \text{(again)} \quad \ddot{\varphi} + \frac{g}{l}\sin\varphi = 0$$

Can we do this in a more general way?

$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_s, t)$ where $q_1, \ldots, q_s$ are generalized coordinates.

$$\delta \vec{r}_i = \sum_{j=1}^{s} \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$ now use d'Alembert:

$$\sum_{i=1}^{n} \left(m_i \ddot{\vec{r}}_i - \vec{F}_i^{(e)}\right) \cdot \delta \vec{r}_i = 0$$

$$\sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j - \underbrace{\sum_{i,j} \vec{F}_i^{(e)} \cdot \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j}_{= \sum_j Q_j \delta q_j} = 0 \qquad (*)$$

$$= \sum_j Q_j\,\delta q_j \quad \text{with} \quad Q_j = \sum_i \vec{F}_i^{(e)} \cdot \frac{\partial \vec{r}}{\partial q_j}$$

In order to handle the first term, use

$$\frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) = \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} + \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

$$= \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} + \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

Consider now $\dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \Rightarrow \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$

This allows to write

$$\ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right) - \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

Back in (*), and we obtain

$$\sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{i,j} m_i \left[\frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right) - \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_j Q_j \delta q_j$$

now introduce $T_i = \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}$

$$\Rightarrow \sum_{i,j} \left(\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{q}_j}\right) - \frac{\partial T_i}{\partial q_j} \right) \delta q_j - \sum_j Q_j \delta q_j = 0$$

Set now $T = \sum_i T_i$ and we find

$$\sum_j \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j}$$

For conservative forces: $\vec{F}_i^{(e)} = -\nabla_i V(\vec{r}_1, \dots, \vec{r}_n)$

$$\Rightarrow Q_j = \sum_i \vec{F}_i^{(e)} \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \frac{\partial V}{\partial \vec{r}_i} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$\Rightarrow$ $L = T - V$ (Lagrange function, or Lagrangian)

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0}$$

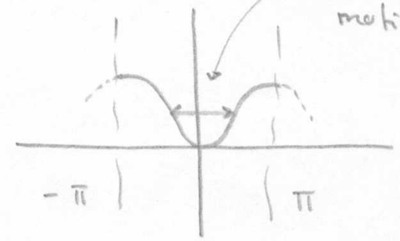
Example: Pendulum again:

$$x = l \sin\varphi \qquad y = l - l\cos\varphi$$

$$T = \frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m l^2 \dot{\varphi}^2$$

$$(\dot{x} = \dot{\varphi} l \cos\varphi \qquad \dot{y} = \dot{\varphi} l \sin\varphi)$$

$$V = mgy = mgl(1-\cos\varphi)$$



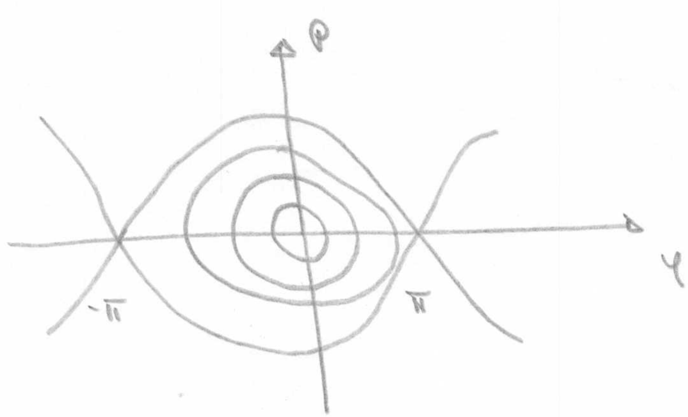
$$L = \frac{1}{2} m l^2 \dot{\varphi}^2 - mgl(1-\cos\varphi)$$

$$\tilde{L} = \frac{1}{2} m l^2 \dot{\varphi}^2 + mgl\cos\varphi$$

$$\frac{\partial \tilde{L}}{\partial \dot{\varphi}} = m l^2 \dot{\varphi} \qquad \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\varphi}} = m l^2 \ddot{\varphi} \qquad \frac{\partial \tilde{L}}{\partial \varphi} = -mgl\sin\varphi$$

Lagrange: $\displaystyle \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\varphi}} - \frac{\partial \tilde{L}}{\partial \varphi} = 0 \Rightarrow m l^2 \ddot{\varphi} + mgl\sin\varphi = 0$

$\Rightarrow \quad \ddot{\varphi} + \frac{g}{l}\sin\varphi = 0$ as before.



phase space

$$E = T + V$$

$$= \frac{1}{2} m l^2 \dot{\varphi}^2 + mlg(1-\cos\varphi)$$

Classification of constraints

Constraints → expressed as equations of coordinates

$f(\vec{r}_1, \ldots, \vec{r}_N, t) = 0$: holonomic constraints

other constraints : nonholonomic (e.g. inequalities)

time as explicit coordinate : theonomous (e.g. bead on wire that is moving)

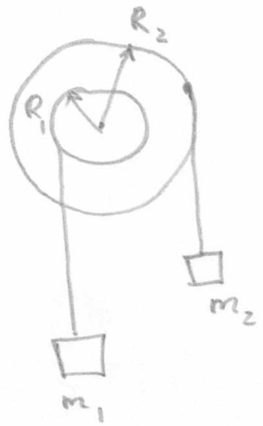
not explicitly time-dependent : scleronomous (e.g. bead on wire that is not moving)

System of N particles : 3N coordinates, k constraints

$\Rightarrow$ 3N − k degrees of freedom.

* Example for Principle of d'Alembert

Atwood's machine : Two point masses on concentric wheels



$$\vec{F}_1^{(c)} \cdot \delta \vec{r}_1 + \vec{F}_2^{(c)} \cdot \delta \vec{r}_2 = 0$$

$$\delta \vec{r}_1 = \delta z_1 \, \vec{e}_z$$

$$\delta \vec{r}_2 = \delta z_2 \, \vec{e}_z$$

$$\delta z_1 = R_1 \delta \varphi = -R_2 \varphi_2$$

The constraining forces can be related to the torque:

$$M_1 = R_1 F_1^{(c)} \qquad M_2 = R_2 F_2^{(c)}$$

$$\vec{F}_i^{(c)} + \vec{F}_i^{(e)} = \dot{\vec{p}}_i \qquad \text{equilibrium:} \quad \dot{\vec{p}}_i = 0$$

$$\Rightarrow \sum_i \vec{F}_i^{(e)} \cdot \delta \vec{r}_i = 0 \qquad m_1 g \, \delta z_1 + m_2 g \, \delta z_2 = 0$$

$$\Rightarrow (m_1 R_1 - m_2 R_2) \, d\varphi = 0$$

$\Rightarrow \quad m_1 R_1 = m_2 R_2$ as condition for equilibrium.