1. Newtonian Mechanics

Fundamental principles: Galieo Galilei (2/15/1564 – 1/8/1642)
Isaac Newton (1/4/1643 – 3/31/1727)

1687 : Philosophiae naturalis principia mathematica
→ groundwork for classical mechanics
→ theory of gravitation → kepler's laws of planetary motion

3 laws of motion:

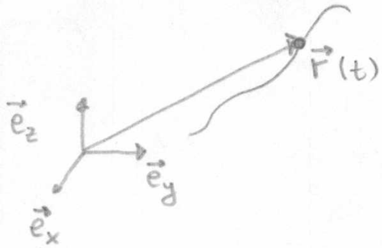
(1) Law of intertia : An object at rest stays at rest, an object in uniform motion stays in uniform motion, unless acted upon by a net external force.

(2) An applied force $\vec{F}$ on an object equals the rate of change of its momentum $\vec{p} = m\vec{v}$.

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \text{and if } m \text{ does not depend on } t \quad \vec{F} = m\frac{d\vec{v}}{dt}$$

$$\vec{F} = m\vec{a}, \quad \vec{a} = \frac{d\vec{v}}{dt} \quad (\text{acceleration})$$

Note, to write this down, we need a coordinate system



$$\vec{r}(t) = x(t)\vec{e}_x + y(t)\vec{e}_y + z(t)\vec{e}_z$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} \equiv \dot{\vec{r}}$$

(3) For every action, there is an equal and opposite reaction.

Note: $\vec{F} = \dot{\vec{p}}$ is only true in an inertial frame (a frame of reference in which a free particle with m travels along a straight line $\vec{r}(t) = \vec{r}_0 + \vec{v}\cdot t$.)

Newton's first law is a statement that such a frame exists.

If $\vec{F} = \vec{F}(\vec{r}, \dot{\vec{r}}, t)$, $m = \text{const}$, we have $m\ddot{\vec{r}} = \vec{F}(\vec{r}, \dot{\vec{r}}, t)$ which is a second-order, ordinary differential equation. Finding the trajectory $\vec{r} = \vec{r}(t)$ corresponds to solving this ODE.

Often: Initial value problem (we know $\vec{r}_0$ and $\vec{v}_0 = \dot{\vec{r}}(0)$).

Consider $\vec{L} = \vec{r} \times \vec{p}$ (angular momentum).

$$\dot{\vec{L}} = \underbrace{\dot{\vec{r}} \times \vec{p}}_{=0} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F} = \vec{M} \quad \text{(called torque)}$$

$\Rightarrow$ $\vec{F} = 0 \Rightarrow \vec{p} = \text{const.}$ $\quad \vec{M} = 0 \Rightarrow \vec{L} = \text{const.}$

Note: $\vec{M} = 0$ is given if $\vec{r} \times \vec{F} = 0$, hence $\vec{F} \parallel \vec{r}$ (central force)

$T = \frac{1}{2} m \dot{\vec{r}}^2$ is called kinetic energy.

$$\frac{dT}{dt} = m \dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{\vec{r}} \cdot \vec{F} \Rightarrow T(t_2) - T(t_1) = \int_{t_1}^{t_2} \vec{F} \cdot \dot{\vec{r}} \, dt$$

$$T(t_2) - T(t_1) = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

$\Rightarrow$ Change of kinetic energy corresponds to work (integral of the force over the path) done by force.

Conservative force: work is independent of path taken.

$$\oint \vec{F} \cdot d\vec{r} = 0 \, , \quad \nabla \times \vec{F} = 0 .$$

For a conservative force, we can find a potential $V = V(\vec{r})$ such that $\vec{F} = -\nabla V(\vec{r}) = -\frac{dV}{d\vec{r}}$

$$\Rightarrow T(t_2) - T(t_1) = \int_{\vec{r}_1}^{\vec{r}_2} -\nabla V \cdot d\vec{r} = -V(\vec{r}_2) + V(\vec{r}_1)$$

$$\Rightarrow T(t_1) + V(\vec{r}_1) = T(t_2) + V(t_2) = E = \text{const.}$$

Conservation of energy.

One-dimensional motion:

We consider $m\ddot{x} = F(x,\dot{x},t)$ with initial conditions $x_0 = x(0)$, $v_0 = v(0)$ given. Special cases:

(a) $F = F(t)$: $\quad v = \dot{x}, \quad \dot{v} = a = \frac{F}{m}$

$$\Rightarrow \quad v(t) = v_0 + \frac{1}{m}\int_0^t F(t')\,dt', \qquad x(t) = x_0 + \int_0^t v(t')\,dt'$$

(b) $F = F(v)$: $\quad \frac{dv}{dt} = \frac{1}{m}F(v) \Rightarrow t = \int_{v_0}^{v} dv' \frac{m}{F(v')} = t(v)$

$v = v(t)$ is the inverse function (that exists in the neighborhood of v^* with $t'(v^*) \neq 0$)

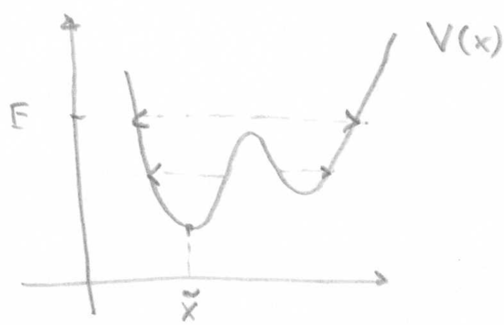
(c) $F = F(x)$: very important and interesting case.

$$m\ddot{x} = F(x) \qquad m\dot{x}\ddot{x} = \dot{x}F(x) \Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{d}{dt}\int_a^{x(t)} F(x')\,dx'$$

Set (again) $T = \frac{1}{2}m\dot{x}^2$, $V = -\int_a^x dx'\,F(x') \Rightarrow T + V = E = \text{const.}$

(Note: a is an arbitrary point of reference.)

As $T \geq 0$, we have $T = E - V \geq 0$. The zeros of $E-V$ are called <u>points of return</u> : $T=0$, hence $v=0$, but usually $F \neq 0 \Rightarrow$ the point mass usually reverses its direction



$$F = -\frac{dV}{dx}$$

The zeros of $\frac{dV}{dx}$ are called <u>points of equilibrium</u>

Stability: $V(x) = V(\tilde{x}) + \underbrace{V'(\tilde{x})(x-\tilde{x})}_{=0} + \frac{1}{2} V''(\tilde{x})(x-\tilde{x})^2 + \dots$

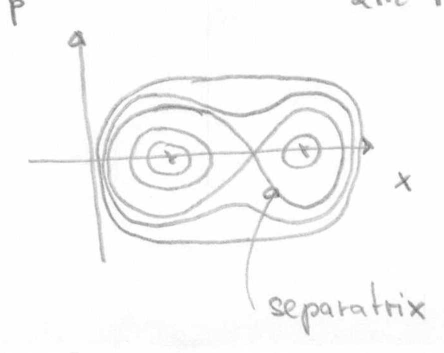
$$\Rightarrow \quad F(x) = -V'(x) \approx -V''(\tilde{x})(x-\tilde{x})$$

Stable equilibrium: $V''(\tilde{x}) > 0$ (force drives mass back to $\tilde{x}$)

unstable equilibrium: $V''(\tilde{x}) < 0$ (force drives mass away from $\tilde{x}$)

<u>Phase plane</u> : $T + V = E = \frac{1}{2} m v^2 + V(x)$ can be written as

$\frac{1}{2m} p^2 + V(x) = E$, this is a curve in (x, p) plane (phase plane)



Analytical Dynamics — Lecture 1

From $\frac{1}{2}mv^2 + V(x) = E$ we find $\frac{dx}{dt} = v = \pm\sqrt{\frac{2}{m}(E - V(x))}$

$$\Rightarrow \quad t(x) = \int_{x_0}^{x} \frac{dx'}{\pm\sqrt{\frac{2}{m}(E-V(x'))}}$$

and we can find the period T between two points of return x_1 and x_2:

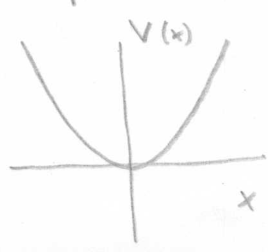
$$T = \int_{x_1}^{x_2} \frac{dx'}{\sqrt{\frac{2}{m}(E-V(x'))}} + \int_{x_2}^{x_1} \frac{dx'}{-\sqrt{\frac{2}{m}(E-V(x'))}} = 2\int_{x_1}^{x_2} \frac{dx'}{\sqrt{\frac{2}{m}(E-V(x'))}}$$

Harmonic oscillator: Consider a deviation from a stable equilibrium at $x = 0$.

$$F(x,v) = \underbrace{F(0,0)}_{(=0)} + \underbrace{\frac{\partial F}{\partial x}}_{\substack{(=-k)\\(k>0)}} x + \underbrace{\frac{\partial F}{\partial v}}_{\substack{(=-2m\beta)\\ \beta>0}} v + \dots$$

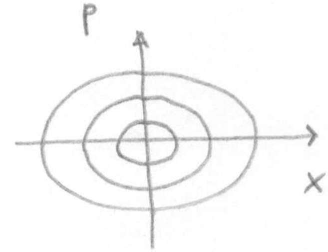
$$m\ddot{x} = -kx - 2m\beta\dot{x} \quad \text{or} \quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \qquad \omega_0^2 = \frac{k}{m}$$

No damping: $\beta = 0 \Rightarrow V(x) = -\int_0^x (-kx')\,dx' = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 x^2$



parabola

$$E = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2$$



phase space

Solution of equation of motion:

$$t = \int_{x_0}^{x} \frac{dx'}{\pm\sqrt{\frac{2}{m}(E - V(x'))}} = \int_{x_0}^{x} \frac{dx'}{\pm\sqrt{\frac{2E}{m} - \omega_0^2 x'^2}} = \int_{x_0}^{x} \frac{dx'}{\pm\omega_0\sqrt{A^2 - x'^2}}$$

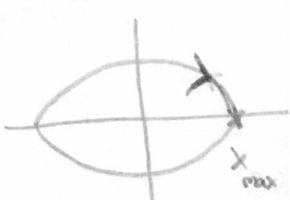
$$A = \frac{2E}{m\omega_0^2} \qquad x' = A\cos\varphi \qquad dx' = -A\sin\varphi\, d\varphi = -\sqrt{A^2 - x'^2}\, d\varphi$$

(in quad. I)

$$\omega_0 t = \int_{x_0}^{x} \frac{-\sqrt{A^2 - x'^2}\, d\varphi}{\sqrt{A^2 - x'^2}} = -\int_{\varphi_0}^{\varphi} d\varphi = \varphi_0 - \varphi \quad \Rightarrow \quad x = A\cos(\varphi_0 - \omega_0 t) = A\cos(\omega_0 t - \varphi_0)$$

for example think of going forwards

to max. x.



$$\Rightarrow \quad T = \frac{2\pi}{\omega_0} \quad \text{is the period}$$

Remark: Phase space: $\dfrac{p^2}{2mE} + \dfrac{m\omega_0^2}{2E} x^2 = 1$ $\qquad a^2 = 2mE$, $\quad b^2 = \dfrac{2E}{m\omega_0^2}$

$\Rightarrow$ ellipse with area $S(E) = \pi a b = \pi \sqrt{2mE}\sqrt{\dfrac{2E}{m\omega_0^2}} = \dfrac{2\pi E}{\omega_0} = TE$

$$\Rightarrow \quad \frac{dS}{dE} = T$$

We can also solve the ODE more directly:

$$\ddot{x} + \omega_0^2 x = 0, \qquad x(t) = A e^{\lambda t}$$

$$\Rightarrow \quad \lambda^2 + \omega_0^2 = (\lambda - i\omega_0)(\lambda + i\omega_0) = 0 \quad \text{(characteristic eq.)}$$

$$\Rightarrow \quad x(t) = A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t}$$

$$v(t) = i\omega_0 (A_1 e^{i\omega_0 t} - A_2 e^{-i\omega_0 t})$$

$$x_0 = A_1 + A_2 \qquad v_0 = i\omega_0 (A_1 - A_2)$$

This method also works with damping:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \qquad x(t) = A e^{\lambda t}$$

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0$$

$$\lambda^2 + 2\beta\lambda + \beta^2 = -\omega_0^2 + \beta^2$$

$$(\lambda + \beta)^2 = -(\omega_0^2 - \beta^2)$$

$$\lambda = \pm i\sqrt{\omega_0^2 - \beta^2} - \beta$$

Solutions: $\underbrace{e^{\pm i\sqrt{\omega_0^2-\beta^2}\,t}}_{\text{oscillations}} \quad \underbrace{e^{-\beta t}}_{\text{damping}}$

Example: $\ddot{x} + \epsilon\dot{x} + x = 0 \quad x(0) = 0, \ \dot{x}(0) = 1, \quad \beta = \frac{\epsilon}{2}$

$$x(t) = A \sin\left(\sqrt{1 - \frac{\epsilon^2}{4}}\, t\right) e^{-\frac{\epsilon}{2} t} \qquad \dot{x}(0) = A\sqrt{1 - \frac{\epsilon^2}{4}} \stackrel{!}{=} 1$$

$$\Rightarrow \quad A = \frac{1}{\sqrt{1 - \epsilon^2/4}}$$

With forcing: we can get resonance !

$$\ddot{x} + x = \cos t \qquad x(0) = 0 \quad \dot{x}(0) = 0$$

frequency of harm. osc. : 1

same frequency

$$x(t) = \frac{1}{2} t \sin t$$

$$\left(\dot{x} = \frac{1}{2}\sin t + \frac{1}{2} t \cos t, \quad \ddot{x} = \frac{1}{2}\cos t + \frac{1}{2}\cos t - \frac{1}{2} t \sin t \right)$$

* Note: This can make it difficult to use asymptotics:

Try to solve $\ddot{x} + \epsilon \dot{x} + x = 0$ using

$$x(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots$$

Leading order: $\ddot{y}_0 + y_0 = 0 \qquad y_0(0) = 0 \quad \dot{y}_0(0) = 1$

$$\Rightarrow y_0(t) = \sin t$$

$O(\epsilon)$: $\ddot{y}_1 + y_1 = -\cos t \qquad y_1(0) = 0 \quad \dot{y}_1(0) = 0$

$$\Rightarrow y_1(t) = -\frac{1}{2} t \sin t \qquad \text{not good !}$$

Fix: multiple time scales

Analytical Dynamics

Try $x(t) = y_0(t_0, t_1) + \epsilon\, y_1(t_0, t_1) + \cdots$

$t_0 = t, \quad t_1 = \epsilon t$

$$\frac{dy_0}{dt} = \frac{\partial y_0}{\partial t_0} + \epsilon \frac{\partial y_0}{\partial t_1}$$

$$\frac{d^2 y_0}{dt^2} = \frac{\partial^2 y_0}{\partial t_0^2} + 2\epsilon \frac{\partial^2 y_0}{\partial t_0 \partial t_1} + \underbrace{\epsilon^2 \frac{\partial^2 y_0}{\partial t_1^2}}_{\mathcal{O}(\epsilon^2)}$$

Leading order: $\dfrac{\partial^2 y_0}{\partial t_0^2} + y_0 = 0 \qquad y_0(0) = 0 \qquad \left.\dfrac{\partial y_0}{\partial t_0}\right|_{t=0} = 1$

$$\left.\begin{aligned} y_0(t) &= A(t_1)\sin(t_0) + B(t_1)\cos(t_0) \\ \frac{\partial y_0}{\partial t_0} &= A(t_1)\cos(t_0) - B(t_1)\sin(t_0) \end{aligned}\right\} \rightarrow \begin{aligned} A(0) &= 1 \\ B(0) &= 0 \end{aligned}$$

$\mathcal{O}(\epsilon)$:

$$\epsilon\left(\frac{\partial^2 y_1}{\partial t_0^2} + y_1\right) = -\epsilon \frac{\partial y_0}{\partial t_0} - 2\epsilon \frac{\partial^2 y_0}{\partial t_0 \partial t_1} = R \cdot \epsilon$$

use dependence on t_1 (slow scale) to "remove" resonances.

$$R = -\left[A(t_1)\cos(t_0) - B(t_1)\sin(t_0)\right]$$
$$-2\left[A'(t_1)\cos(t_0) - B'(t_1)\sin(t_0)\right]$$

$\Rightarrow$ $-A - 2A' = 0$, $\quad B + 2B' = 0$

$A(t_1) = A(0)\, e^{-t_1/2}$ $\quad B(t_1) = B(0)\, e^{-t_1/2}$

$\Rightarrow$ $y_0(t_0, t_1) = e^{-t_1/2} \sin(t_0)$ or $X(t) \approx e^{-\frac{\epsilon t}{2}} \sin(t)$

as approximation