Due: Nov 14 2024 **MTH 339 Algebra Homework 6**

1. (10 pts) Let $G = \mathrm{GL}(2, \mathbb{R})$ and let K be a subgroup of $\mathbb{R}^*$. Prove that $H = \{A \in G \mid \det(A) \in K\}$ is a normal subgroup of G.

2. (10 pts) Prove that a factor group of an Abelian group is Abelian.

3. (10 pts) In $\mathbb{Q}/\langle 3.5\rangle$ find the unique element $a+\langle 3.5\rangle$ such that $a+\langle 3.5\rangle = 8+\langle 3.5\rangle$ where $0 < a < 3.5$.

4. (10 pts) Let H be a subgroup of a group G with the property that for all a and b in G, $aHbH = abH$. Prove that H is a normal subgroup of G.

5. (10 pts) Let N be a normal subgroup of G. Prove that if N is cyclic, every subgroup of N is also normal in G.

6. (10 pts) Suppose that ϕ is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ and $\operatorname{Ker}\phi = \{0, 10, 20\}$. If $\phi(23) = 9$, determine all elements that map to 9.

7. (10 pts) Prove that there is no homomorphism from $\mathbb{Z}_8 \bigoplus \mathbb{Z}_2$ onto $\mathbb{Z}_4 \bigoplus \mathbb{Z}_4$.

8. (10 pts) Suppose that ϕ is a homomorphism from $U(30)$ to $U(30)$ and that $\operatorname{Ker}\phi = \{1, 11\}$. If $\phi(7) = 7$, find all elements that map to 7.

9. (10 pts) Suppose that there is a homomorphism from a finite group G onto $\mathbb{Z}_{10}$. Prove that G has normal subgroups of indexes 2 and 5.

10. (10 pts) If K is a subgroup of G and N is a normal subgroup of G, prove that $K/(K \cap N)$ is isomorphic to KN/N.