Normal Subgroups

Def A subgroup H of a group G is called a normal subgroup of G if $aH = Ha$ for all $a \in G$. We denote this by $H \triangleleft G$.

Thm A subgroup H of G is normal in G if and only if $xHx^{-1} \subseteq H$ for all x in G.

Proof: If H is normal in G, then for any $x \in G$ and $h \in H$, there is an $h' \in H$ such that $xh = h'x$. Thus $xhx^{-1} = h'$ and, therefore, $xHx^{-1} \subseteq H$.
Conversely, if $xHx^{-1} \subseteq H$, letting $x = a$, we have $aHa^{-1} \subseteq H$ or $aH \subseteq Ha$.
On the other hand, letting $x = a^{-1}$, we have $a^{-1}H(a^{-1})^{-1} = a^{-1}Ha \subseteq H$ or $Ha \subseteq aH$.

$\square$

Examples:

- Every subgroup of an Abelian group is normal, since $ah = ha$ for $a \in G$ and $h \in H$.
- The center $Z(G)$ is always normal. We have $ah = ha$ for any $a \in G$ and any $h \in Z(G)$.

Example:

- Let H be a normal subgroup of a group G and K be any subgroup of G. Then
  $$HK = \{ hk \mid h \in H, k \in K \}$$
  is a subgroup of G. ($\rightarrow$ Do now)

Example: $SL(2, \mathbb{R})$ (group of 2×2 matrices with determinant 1) is a normal subgroup of $GL(2, \mathbb{R})$ (group of 2×2 matrices with nonzero determinant). ($\rightarrow$ Do now)

Factor Groups:

Thm Let G be a group and let H be a normal subgroup of G. The set $G/H = \{aH \mid a \in G\}$ is a group under the operation $(aH)(bH) = abH$

Proof: ① We show that the operation is well-defined. In particular, we need to make sure that the operation depends only on the cosets and not on the particular choice of the coset representatives.

In other words, if $aH = a'H$ and $bH = b'H$, we need to show that $aH\,bH = a'H\,b'H$, or - equivalently - $abH = a'b'H$.

Since $aH = a'H$ and $bH = b'H$, we have $a' = ah_1$ and $b' = bh_2$ for some $h_1, h_2 \in H$ and, therefore $a'b'H = ah_1 bh_2 H = ah_1 bH$

$= ah_1 Hb = aHb = abH$

(H is normal) (H is normal)

② Now we prove the remaining properties. $eH = H$ is the identity, $a^{-1}H$ is the inverse of aH. And it is easy to see that

$$(aH\, bH)\, cH = (abH)\, cH = (ab)cH$$

$$= a(bc)H = aH\,(bc)H = aH\,(bH\, cH)$$

□

Example: Consider $4\mathbb{Z} = \{0, \pm 4, \pm 8, \dots\}$

To construct $\mathbb{Z}/4\mathbb{Z}$, we need the left cosets:

$$0 + 4\mathbb{Z} = \{0, \pm 4, \pm 8, \dots\}$$
$$1 + 4\mathbb{Z} = \{1, 5, 9, \dots ; -3, -7, -11, \dots\}$$
$$2 + 4\mathbb{Z} = \{2, 6, 10, \dots ; -2, -6, -10, \dots\}$$
$$3 + 4\mathbb{Z} = \{3, 7, 11, \dots ; -1, -5, -9, \dots\}$$

Cayley-table ($\rightarrow$ Do now)

	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$0+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$
$1+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$	$0+4\mathbb{Z}$
$2+4\mathbb{Z}$	$2+4\mathbb{Z}$	$3+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$
$3+4\mathbb{Z}$	$3+4\mathbb{Z}$	$0+4\mathbb{Z}$	$1+4\mathbb{Z}$	$2+4\mathbb{Z}$

So $\mathbb{Z}/4\mathbb{Z} \approx \mathbb{Z}_4$

<u>Note:</u> When we factor out by a normal subgroup H, we are essentially defining any element of H to be the identity.

In $\mathbb{Z}/4\mathbb{Z}$ any multiple of 4 is the identity: $5 + 4\mathbb{Z} = 1 + 4 + 4\mathbb{Z} = 1 + 4\mathbb{Z}$.

<u>Note:</u> The structure of G/H is usually less complicated than the structure of G and can help us to analyze the stucture of G.

Thm G/Z Theorem

Let G be a group and let $Z(G)$ be the center of G. If $G/Z(G)$ is cyclic, then G is Abelian.

<u>Proof:</u> Since the statement that G is Abelian is equivalent to $Z(G) = G$, we need to show that the only

element of $G/Z(G)$ is the identity coset $Z(G)$.

Let $G/Z(G) = \langle gZ(G) \rangle$ and $a \in G$. Then, there exists an integer i such that $aZ(G) = (gZ(G))^i = g^i Z(G)$, so $a = g^i z$ for some $z \in Z(G)$.

Since both g^i and z belong to $C(g)$, we have $a \in C(g)$. But a is arbitrary, so all $a \in G$ commute with the element g, so $g \in Z(G)$, so $gZ(G) = Z(G)$

□

MTH 339 - Do now

1./ Prove that HK is indeed a subgroup if K is a subgroup and H is a normal subgroup.

2./ Prove that $SL(2,\mathbb{R})$ is a normal subgroup of $GL(2,\mathbb{R})$.

3./ Construct the Cayley table for $\mathbb{Z}/4\mathbb{Z}$.

MTH 339 - HW

1. Let $G = GL(2, \mathbb{R})$ and let k be a subgroup of $\mathbb{R}^*$. Prove that $H = \{A \in G \mid \det A \in k\}$ is a normal subgroup of G.

2. Prove that a factor group of an Abelian group is Abelian.

3. In $\mathbb{Q}/\langle 3.5 \rangle$ find the unique element $a + \langle 3.5 \rangle$ such that $a + \langle 3.5 \rangle = 8 + \langle 3.5 \rangle$ where $0 < a < 3.5$.

4. Let H be a subgroup of a group G with the property that for all a and b in G, $aHbH = abH$. Prove that H is a normal subgroup of G

5. Let N be a normal subgroup of G. Prove that if N is cyclic, every subgroup of N is also normal in G.