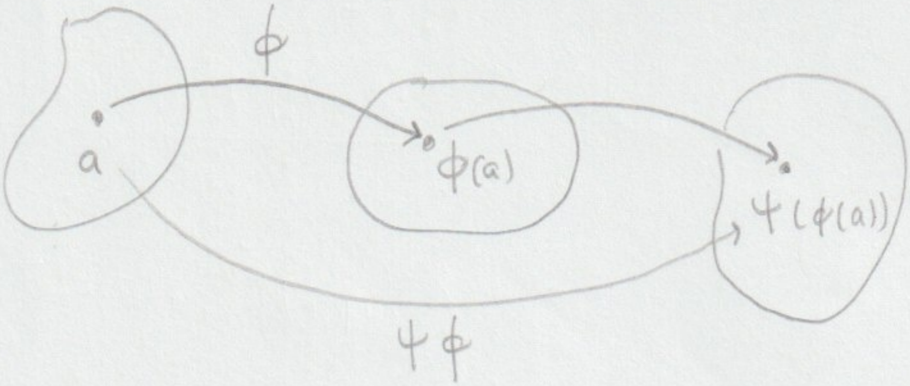
## Functions (Mappings)

**Def** Function (Mapping)

A function $\phi$ (or mapping) from a set $A$ to a set $B$ is a rule that assigns to each element of $A$ exactly one element of $B$. The set $A$ is called the domain of $\phi$ and $B$ is called the codomain of $\phi$. If $\phi$ assigns $b$ to $a$, then $b$ is called the image of $a$ under $\phi$. The subset of $B$ comprised of all the images of elements of $A$ is called the image of $A$ under $\phi$. We write $\phi : A \to B$ and $\phi(a) = b$.

**Def** Composition of Functions

Let $\phi : A \to B$ and $\psi : B \to C$. The composition $\psi\phi$ is the mapping from $A$ to $C$ defined by $(\psi\phi)(a) = \psi(\phi(a))$ for all $a$ in $A$.




Def

One-to-One Function

$\phi: A \to B$ is called one-to-one if

$\phi(a_1) = \phi(a_2)$ implies $a_1 = a_2$

Def

Function from A onto B

A function from A to B is said to be onto B if each element of B is the image of at least one element of A.

Example:

| Domain | Codomain | Rule | 1-1 | Onto |
|---|---|---|---|---|
| $\mathbb{Z}$ | $\mathbb{Z}$ | $x \to x^3$ | yes | no |
| $\mathbb{R}$ | $\mathbb{R}$ | $x \to x^3$ | yes | yes |
| $\mathbb{Z}$ | $\mathbb{N}$ | $x \to \lvert x \rvert$ | no | yes |
| $\mathbb{Z}$ | $\mathbb{Z}$ | $x \to x^2$ | no | no |

$\boxed{\text{MTH 339}}$ 3/6

## Symmetries of a Square:

P – Pink W – White
G – Green B – Blue

$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{R_0} \begin{bmatrix} P & W \\ G & B \end{bmatrix}$ $R_0$ (rotation of $0^\circ$)

$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{R_{90}} \begin{bmatrix} W & B \\ P & G \end{bmatrix}$ $R_{90}$ (rotation of $90^\circ$, counter clockwise)

$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{R_{180}} \begin{bmatrix} B & G \\ W & P \end{bmatrix}$ $R_{180}$ (rotation of $180^\circ$, counterclockwise)

$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{R_{270}} \begin{bmatrix} G & P \\ B & W \end{bmatrix}$ $R_{270}$ (rotation of $270^\circ$, counter clockwise)

$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{H} \begin{bmatrix} G & B \\ P & W \end{bmatrix}$ $H$ (rotation of $180^\circ$ about a horizontal axis)

$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{V} \begin{bmatrix} W & P \\ B & G \end{bmatrix}$ $V$ (rotation of $180^\circ$ about a vertical axis)

$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{D} \begin{bmatrix} P & G \\ W & B \end{bmatrix}$ $D$ (rotation of $180^\circ$ about the main diagonal)

$\begin{bmatrix} P & W \\ G & B \end{bmatrix} \xrightarrow{D'} \begin{bmatrix} B & W \\ G & P \end{bmatrix}$ $D'$ (rotation of $180^\circ$ about the other diagonal)

$\boxed{\text{MTH 339}}$

4/6

**Remark:** Any motion, no matter how complicated, is equivalent to one of these eight.

**Reason:** Final position of corner: 4
Final orientation: 2
$\hookrightarrow$ total of 8 positions

**Example:** $D = H R_{90}$

$$\begin{array}{|c|c|}\hline P & W \\ \hline G & B \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline W & B \\ \hline P & G \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline P & G \\ \hline W & B \\ \hline\end{array}$$

**Remark:** It is possible to construct a complete operation table (Cayley table).

The eight motions, together with composition, are called dihedral group of order 8, or $D_4$.

**Observation:** $D_4$ is <u>closed</u> under composition. Some elements have inverse elements. $HH = D_0$, and $D_0$ is the neutral element. Some elements don't commute, e.g. $HD \neq DH$, but $R_{90} R_{180} = R_{180} R_{90}$.

## Definition and Examples of Groups

Def **Binary Operation**

Let $G$ be a set. A binary operation on $G$ is a function that assigns each ordered pair of elements of $G$ an element of $G$.

$\phi(R_{90}, H) = HR_{90} = D$ as an example.

Other examples:

- addition, subtraction, multiplication of integers
- binary operations addition modulo $n$ and multiplication modulo $n$ on $\{0, 1, 2, .. n-1\}$ $\mathbb{Z}_n$.

Def **Group:**

$G$ nonempty set with binary operation (usually called multiplication)

We say $G$ is a <u>group</u> under this operation if the following three properties are satisfied

1. Associativity $(ab)c = a(bc) \quad \forall\, a,b,c \in G$
2. Identity: $\exists\, e \in G : ae = ea = a, \quad \forall\, a \in G$
3. Inverses: $\forall\, a \in G \ \exists\, b \in G : ab = ba = e$

<u>Note:</u> If $\forall\, a,b : ab = ba$, we say the group is <u>Abelian</u>. A group is <u>non-Abelian</u> if there is a pair for which $ab \neq ba$.

<u>Example</u>: $\mathbb{Z}$ (set of integers), $\mathbb{Q}$, $\mathbb{R}$ are groups under ordinary addition. The identity is $0$ and for any $a$, the inverse is $-a$.

MTH 339 - Do Now

1./ Let $\phi(x) = \sqrt{x-3}$. Find the maximum domain $A \subset \mathbb{R}$. Let $B = \phi(A)$. Is $\phi$ one-to-one?

2./ Show that $\phi : \mathbb{R} \to \mathbb{R}$ $\phi(x) = x^2 + 2$ is not one-to-one.

3./ Show $H R_{180} = V$, $HD \neq DH$,

$R_{90} R_{180} = R_{180} R_{90}$

# MTH 339 - HW

1./ With pictures and words, discribe each symmetry in $D_3$.

2./ Write out a complete multiplication table for $D_3$.

3./ Is $D_3$ Abelian?

4./ Find elements in $D_4$ such that $AB = BC$, but $A \neq C$.

5./ Describe the symmetries of a nonsquare rectangle. Construct the corresponding Cayley table.