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Mathematical Induction:

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First Principle: Let S be a set of integers containing a. Suppose S has the property that whenever some integer $n \geq a$ belongs to S, then the integer $n+1$ also belongs to S. Then, S contains every integer greater or equal to a.

Ex: For all $n \in \mathbb{N}$, $3 \mid 2^{2n} - 1$

Proof:

$\boxed{n=1}$: $2^{2n} - 1 = 2^2 - 1 = 3$

$\boxed{n \to n+1}$: Assume $3 \mid 2^{2n} - 1$. We must show $2^{2(n+1)} - 1$ is divisible by 3.

$$2^{2(n+1)} - 1 = 4 \cdot 2^{2n} - 1 = 4 \cdot 2^{2n} - 4 + 3$$
$$= 4\,(2^{2n} - 1) + 3$$

which is divisible by 3. $\square$

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Review: Complex numbers:

Motivation: $x^2+1=0$ does not have a solution in $\mathbb{R}$

$\hookrightarrow$ set $i=\sqrt{-1}$ (imaginary unit)

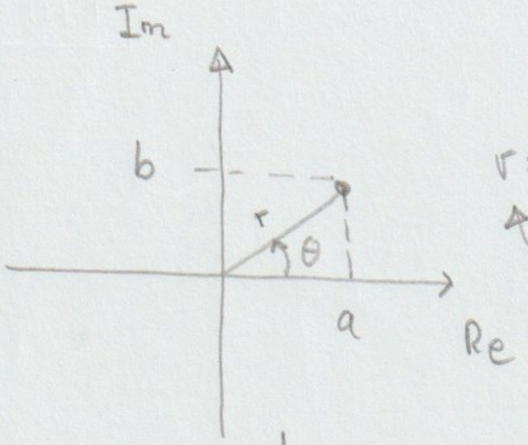
$$\mathbb{C}=\{z \mid z=a+bi \text{ with } a,b\in\mathbb{R}\}$$

is the set of complex numbers.

Remark: Addition, subtraction, and multiplication is straightforward $(i^2=-1)$

$2+3i+4-2i=6-i$, $(2+3i)(4-2i)$

$= 8-4i+12i-6i^2 = 14-8i$



$r=|z|=\sqrt{a^2+b^2}$ $\quad z=re^{i\theta}$

(distance from origin on complex plane)

$\tan\theta=\frac{b}{a}$

$|2+3i|=\sqrt{4+9}=\sqrt{13}$

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(Def) complex conjugate

$\bar{z} = a - bi$ if $z = a + bi$

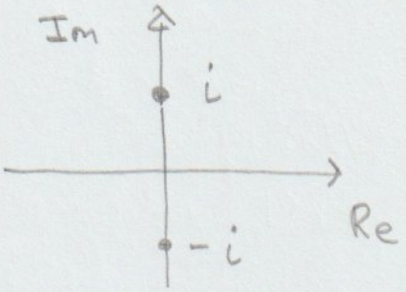
$$z\bar{z} = (a-bi)(a+bi) = a^2 + b^2 = |z|^2$$

Division: $z^{-1} = \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$

So $$\frac{2+3i}{4-2i} = \frac{(2+3i)(4+2i)}{(4-2i)(4+2i)} = \frac{2+16i}{20}$$

$$= \frac{1}{10} + \frac{4}{5}i$$

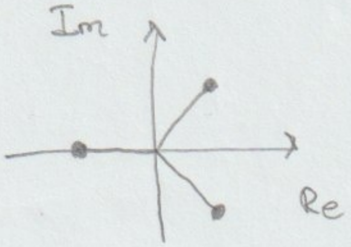
Computing roots: $z^2 = -1 \Rightarrow z = i$ or $z = -i$



Polar coordinates $z^3 = -1$ $\quad r^3 e^{3i\theta} = -1$

$\Rightarrow r = 1$ and $e^{3i\theta} = \cos(3\theta) + i\sin(3\theta) = -1$

$\theta = \frac{\pi}{3}$, $\theta = \pi$, $\theta = -\frac{\pi}{3}$



Recall: $e^{ix} = \cos(x) + i\sin(x)$

(Euler's formula)

Example: DeMoivre's Theorem

For all positive integers n and all $\theta \in \mathbb{R}$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

$n=1$: trivial

$n \to n+1$ $\quad (\cos\theta + i\sin\theta)^{n+1}$

$$= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta)$$

$$= (\cos(n\theta) + i\sin(n\theta))(\cos\theta + i\sin\theta)$$

$$= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin\theta$$

$$+ i(\sin(n\theta)\cos\theta + \cos(n\theta)\sin\theta)$$

$$= \cos((n+1)\theta)) + i\sin((n+1)\theta)$$

$\square$

Equivalence Relations:

Def An equivalence relation on a set S is a set R of ordered pairs of S such that

(1) $(a,a) \in R$ for all $a \in S$

(2) $(a,b) \in R \Rightarrow (b,a) \in R$ (symmetric)

(3) $(a,b) \in R$ and $(b,c) \in R$
$\Rightarrow (a,c) \in R$ (transitive)

When R is an equivalence relation, we often write aRb instead of $(a,b) \in R$, or $a \sim b$.

The set $[a] = \{x \in S \mid x \sim a\}$ is called the equivalence class of S containing a.

Example: S = set of all triangles in a plane
$a, b \in S$: $a \sim b$ if a and b are similar.
$\sim$ is an equivalence relation on S.

Example: Let $S = \mathbb{Z}$ and n be a positive integer. For $a, b \in S$ define $a \equiv b$ if $a = b \mod n$ $(n \mid (a-b))$

Then "$\equiv$" is an equivalence relation.

Proof: $[a] = \{a + kn \mid k \in \mathbb{Z}\}$

(1) $a - a$ is divisible by n, so $a \equiv a$.

(2) Assume $a - b$ is divisible by n,

$a - b = kn \Rightarrow b - a = -kn \Rightarrow b \equiv a$.

(3) $a \equiv b$ and $b \equiv c \Rightarrow a - b = rn$ and $b - c = sn \Rightarrow a - c = (r+s)n$ $\square$

Example: Let $n = 7$, "$\equiv$" as above. Then

$16 \equiv 2$ $(16 - 2 = 14$ is divisible by $7)$

$9 \equiv -5$

$24 \equiv 3$

What would $[1]$ be? For $\xi \in [1]$

$1 = \xi \mod 7$ or $1 - \xi = 7k$

$[1] = \{ \ldots -20, -13, -6, 1, 8, 15 \ldots \}$

Def : Partition

A *partition* of a set S is a collection of nonempty disjoint subsets of S whose union is S.

Ex: $A = \{0\}$

$B = \{1, 2, 3, \dots\}$ $C = \{\dots, -3, -2, -1\}$

Then A, B, C are disjoint subsets of $\mathbb{Z}$ and $\mathbb{Z} = A \cup B \cup C$.

Th Equivalence Classes Partition

The equivalence classes of an equivalence relation on a set S constitute a partition of S. Conversely, for any partition $\mathcal{P}$ of S, there is an equivalence relation on S whose equivalence classes are the elements of $\mathcal{P}$.

Proof:

"⇒" Let $\sim$ be an equivalence relation on S.
For any $a \in S$, $a \sim a$ so $[a] \neq \emptyset$.
Clearly, the union of all equivalence classes is S. We only need to show that for two different classes, their intersection is empty.
Assume $c \in [a] \cap [b]$ and we show $[a] = [b]$.
It is sufficient to prove $[a] \subseteq [b]$.
Let $x \in [a]$. We have $c \sim a$, $c \sim b$, and $x \sim a$. Symmetry implies $a \sim c$. Then we use transitivity: $x \sim a$ and $a \sim c$ $\Rightarrow x \sim c$ and from $x \sim c$ and $c \sim b$ we find $x \sim b$, so $x \in [b]$.

"⇐": Exercise.

MTH 339 - Do now

1./ Compute: $z_1 = 3+4i$, $z_2 = 2-i$

(a) $z_1 + z_2$, $z_1 - z_2$, $z_1 \cdot z_2$, $\frac{z_1}{z_2}$

(b) $\overline{z_1}\,\overline{z_2}$, $|z_1|$, r, θ in $z_2 = re^{i\theta}$

(c) Solve $z^2 = z_2$

2./ Use induction to prove $\sum_{k=1}^{n} k = \frac{n}{2}(n+1)$

3./ Let $a \equiv b$ if $a \equiv b \mod 7$. Find $[4]$.

MTH 339 - HW

1./ Fibonacci numbers: 1, 1, 2, 3, 5, 8 ...

$f_1 = 1$, $f_2 = 1$, $f_{n+2} = f_{n+1} + f_n \quad n = 1, 2, 3, \ldots$

Prove that $f_n < 2^n$.

2./ Prove that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3./ Review the Taylor Theorem (Calc II) and use Taylor series to prove

$$e^{ix} = \cos(x) + i\sin(x)$$

4./ Complete the proof of the equivalence classes partition theorem

5./ Let $S = \{(x,y) \mid x, y \in \mathbb{R}\}$. Define $\sim$ by $(a,b) \sim (c,d)$ if $a^2+b^2 = c^2+d^2$. Prove that $\sim$ is an equivalence relation on S. Give a geometric interpretation.