MTH 416 - Fall 2019 - Practice II
Review: Mon Dec 2, 2019

1. Let W_t be a $\mathbb{P}-$Brownian motion. and let $Y_t = W_t + t$.
   (a) What are $\mathbb{E}_{\mathbb{P}}(Y_t)$ and $\mathbb{E}_{\mathbb{P}}(Y_t^2)$?
   (b) Find $\mathbb{E}_{\mathbb{P}}(Y_t^3)$. Hint: Make use of the symmetries of the underlying probability distribution.
   (c) Use Ito's Lemma to find $d(Y_t^2)$ and $d(Y_t^3)$.

2. Consider $X_t = \sigma W_t + \mu t$ with a $\mathbb{P}$-Brownian motion and use the Girsanov theorem to find a $\mathbb{Q}$- Brownian motion $\tilde{W}_t$ such that $X_t = \sigma \tilde{W}_t$. Compute both $\mathbb{E}_{\mathbb{Q}}(X_t^2)$ and $\mathbb{E}_{\mathbb{P}}(X_t^2)$.

3. Let $S_t = \exp(\mu t + \sigma W_t)$ with a $\mathbb{P}$-Brownian motion W_t. Consider the random process $X_t = \exp(-rt)S_t$.
   (a) Write down the stochastic differential equation that X_t satisfies.
   (b) Use the Girsanov theorem to construct a measure $\mathbb{Q}$ and a $\mathbb{Q}$-Brownian motion $\tilde{W}_t$ such that dX_t is driftless with respect to $\mathbb{Q}$.
   (c) Write S_t in terms of $\tilde{W}_t$.

4. Let $X = S_t^{-\alpha}$, with
$$S = S_0\, \mathrm{e}^{\sigma W_t + (r - \sigma^2/2)t}$$
with a $\mathbb{Q}$-Brownian motion W_t. For which α is X a $\mathbb{Q}$-martingale?

5. Consider a stock process that follows an exponential Brownian motion under the measure $\mathbb{P}$
$$S_t = S_0 \mathrm{e}^{\sigma W_t + \mu t} \tag{1}$$
We know that under the risk-free measure $\mathbb{Q}$, using a $\mathbb{Q}$-Brownian motion $\tilde{W}_t$, we can write
$$S_t = S_0 \mathrm{e}^{\sigma \tilde{W}_t + (r - \sigma^2/2)t} \tag{2}$$
where r is the interest rate. Assume now $r = 5\%$, $\mu = 10\%$, $\sigma = 30\%$ and consider a claim that pays \$100 if, after two years, the stock price S_T is larger than $2 \cdot S_0$, if $S_0 = \$100$ is the initial value of the stock. What is the price of such a claim?