Problem 1 (10 points)

Let W_t be a $\mathbb{P}$-Brownian motion.

1. Compute $\mathbb{E}_{\mathbb{P}}(W_t^2 - 2t)$
2. Compute $\mathbb{E}_{\mathbb{P}}(\mathrm{e}^{2W_t})$
3. Assume that
$$p(x,t) = \alpha(t)x^2\,\mathrm{e}^{-x^2/(2t)}$$
is the density of a process X_t. Find $\alpha(t)$. Hint: $\mathbb{E}(W_t^2) = t$.

Problem 2 (10 points)

Let W_t be a $\mathbb{P}$-Brownian motion.

1. For $X_t = \mathrm{e}^{-2W_t}$, compute dX_t.
2. For $X_t = W_t^3 - \mathrm{e}^{2t}W_t$, compute dX_t.
3. Which equation do σ and μ satisfy if
$$X_t = (\sigma W_t)^2 - \mu t$$
is a martingale?

Problem 3 (10 points)

Consider a stock $S_t = S_0\mathrm{e}^{\sigma W_t + t}$, with a $\mathbb{P}$-Brownian motion W_t.

1. Show that S_t is not a $\mathbb{P}$-martingale.
2. Use the Girsanov theorem to construct a measure $\mathbb{Q}$ such that S_t is a $\mathbb{Q}$-martingale. Express S_t in terms of a $\mathbb{Q}$-Brownian motion $\tilde{W}_t$.
3. Assume $\sigma = 0.3$, $S_0 = \$10$. No interest rates. What is the value of a bet that pays \$20 if the stock is worth more than \$9 after two years?