Binomial Tree - Summary

$$\begin{aligned} q &= \frac{e^{r\delta t}s_{\text{now}} - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}} \\ f_{\text{now}} &= e^{-r\delta t}\left(qf_{\text{up}} + (1-q)f_{\text{down}}\right) \\ V &= f(1) = \mathbb{E}_{\mathbb{Q}}\left(B_T^{-1}X\right) \end{aligned} \qquad \begin{aligned} \phi &= \frac{f_{\text{up}} - f_{\text{down}}}{s_{\text{up}} - s_{\text{down}}} \\ \psi &= B_{\text{now}}^{-1}\left(f_{\text{now}} - \phi s_{\text{now}}\right) \end{aligned}$$

where

q: arbitrage probability of up-jump
f: claim value time-process
ϕ: stock holding strategy
ψ: bond holding strategy
V: claim value at time zero
δt: length of period

r: interest rate in force over period
s: stock price process
B: bond price process, $B_0 = 1$
$\mathbb{Q}$: measure made up of the qs
X: claim payoff
T: time of claim payoff

Black-Scholes formula - (European call option)

$$\begin{aligned} V &= \mathbb{E}_{\mathbb{Q}}\left[\left(S_0 \exp(\sigma\sqrt{T}Z - \frac{\sigma^2}{2}T) - k\exp(-rT)\right)^+\right] \\ &= S_0\Phi\left(\frac{\log(S_0/k) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - ke^{-rT}\Phi\left(\frac{\log(S_0/k) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \end{aligned}$$

Itô's Lemma

If X is a stochastic process, satisfying

$$dX_t = \sigma_t dW_t + \mu_t dt$$

and f is a deterministic twice continuously differentiable function, then $Y_t = f(X_t)$ is also a stochastic process and is given by

$$dY_t = \left(\sigma_t f'(X_t)\right) dW_t + \left(\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t)\right) dt.$$