Make-up Class

Topics : (a) Pricing of a put-option using the Black-Scholes model

(b) Put-Call Parity

(c) More about the "Greeks"

(a) Recall that in the Black-Scholes setting, a stock is modeled as

$$S_t = S_0 \, e^{\mu t + \sigma W_t}$$

and the bond is given by $B_t = B_0 e^{rt}$, $B_0 = 1$

For risk-neutral pricing, we form the discounted stock process $Z_t = e^{-rt} S_t = S_0 \, e^{\sigma W_t + (\mu - r)t}$ and use the Girsanov-theorem to construct the martingale measure $\mathbb{Q}$. Let's review these steps:

$$dZ_t = Z_t \left(\sigma \, dW_t + \left(\mu - r + \frac{\sigma^2}{2}\right) dt \right)$$

follows directly from Ito's Lemma. Girsanov:

$$dZ_t = \sigma Z_t \, (\underbrace{dW_t + \frac{1}{\sigma}\left(\mu - r + \frac{\sigma^2}{2}\right) dt}_{d\tilde{W}_t = dW_t + \gamma dt})$$

set $\gamma = \frac{1}{\sigma}\left(\mu - r + \frac{\sigma^2}{2}\right)$ $\qquad d\tilde{W}_t = dW_t + \gamma dt$

and we obtain $dZ_t = \sigma Z_t \, d\tilde{W}_t$ using the $\mathbb{Q}$-B.M. $\tilde{W}_t$.

Thus $Z_t = S_0 e^{\sigma \tilde{W}_t - \frac{\sigma^2}{2} t}$ and finally

$$S_t = S_0 e^{\sigma \tilde{W}_t + (r - \frac{\sigma^2}{2}) t}$$

is the stock process expressed via the $\mathbb{Q}$-B.M. $\tilde{W}_t$. Any claim X can be valued using $V = e^{-rT} \mathbb{E}_{\mathbb{Q}}(X)$.

For a European put option, we have $X = (K - S_T)^+$ and

$$V = e^{-rT} \mathbb{E}_{\mathbb{Q}}\left(\left(K - S_0 e^{\sigma \tilde{W}_T + (r - \frac{\sigma^2}{2}) T}\right)^+\right)$$

$$= \mathbb{E}_{\mathbb{Q}}\left(\left(K e^{-rT} - S_0 e^{\sigma \tilde{W}_T - \frac{\sigma^2}{2} T}\right)^+\right)$$

we can now proceed exactly the same way as for a European call option

$$V = \int_{-\infty}^{z_c} \left(K e^{-rT} - S_0 e^{\sigma \sqrt{T} z - \frac{\sigma^2}{2} T}\right)^+ \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

where z_c is given by $e^{-rT} = S_0 e^{\sigma \sqrt{T} z_c - \frac{\sigma^2}{2} T}$

After some algebra (ex I), we obtain the Black-Scholes formula for the European put:

Black-Scholes price of a European put option:

$$V = e^{-rT} \cdot K \cdot \Phi(-d_2) - S_0\, \Phi(-d_1) \qquad (*)$$

with $d_1 = \dfrac{\ln(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$ and

$$d_2 = \frac{\ln(S_0/K) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

(b) There is a different way to obtain (*) from the price of a call option by using the so-called "put-call parity".
Consider two portfolios:

(a) one European call + cash equal to Ke^{-rT}
(b) one European put + one share of the stock.

at expiration T, both portfolios are worth

$$\max(S_T, K)$$

(to verify this, consider the scenarios $S_T > K$ and $S_T < K$ for both portfolios)
Since our market is assumed to be arbitrage-free, the portfolios must have the same values today,

hence $\underbrace{C}_{\text{call price}} + Ke^{-rT} = \underbrace{P}_{\text{put price}} + S_0$

This yields a different way to derive (*) for the Black-Scholes model (ex II)

$$\mathcal{P} = C + Ke^{-rT} - S_0$$

(c) More about the "Greeks".

We already know that, for a European call in the Black-Scholes model, the required holding of the stock is

$$\Delta = \Phi(d_1) \quad \text{with}$$

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

For a put option, it follows immediately from put-call parity that $\Delta = N(d_1) - 1$, meaning that Δ is actually negative.

Δ measures the sensitivity of the option price with respect to change of the stock value

$$\Delta = \frac{\partial V}{\partial S}$$

We can define other sensitivities, e.g. vega (or kappa $\varkappa$), which is the rate of change of the option price with respect to the volatility σ.

$$\varkappa = \frac{\partial V}{\partial \sigma} = S_t \sqrt{T-t}\,\Phi'(d_1) \qquad (\text{see HW } 6)$$

Other "Greeks" are

$$\left.\begin{aligned} \varrho &= \frac{\partial V}{\partial r} = K\,(T-t)\,e^{-r(T-t)}\,\Phi(d_2) \\ \Gamma &= \frac{\partial^2 V}{\partial S^2} = \frac{\Phi'(d_1)}{S_t\,\sigma\sqrt{T-t}} \end{aligned}\right\} \quad (**)$$

<u>Exercises</u>: (Each worth 10 points)

(I) Prove (*), the Black-Scholes formula for a European put-option by direct calculation

(II) Prove (*) from put-call parity

(III) Derive the expressions for ϱ and Γ given in (**) for a call option.