Construction Strategies

The martingale representation theorem in the continuous world: Let's for a moment look again at the simple case of zero-interest rates. Remember that, in the discrete world, the binomial representation theorem would allow us to construct a self-financing hedging strategy to replicate our claim: First, we construct a measure $\mathbb{Q}$, such that the stock process S on the tree is a $\mathbb{Q}$-martingale. Then, in the second step, we convert the claim X into a process E, using the conditional expectation operator $E_i = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_i)$. At each step, we have

$$\Delta E_i = \phi_i \Delta S_i, \qquad E_i = E_0 + \sum_{k=1}^{i} \phi_k \Delta S_k \,.$$

In other words, at time-tick i, we need ϕ_{i+1} units of the stock S and, therefore, $\psi_{i+1} = E_i - \phi_{i+1} S_i$ units of the bond. At time zero, our portfolio has the value $\phi_1 S_0 + \psi_1 = E_0 = \mathbb{E}_{\mathbb{Q}}(X)$ which is the money we need to create it (the price of the derivative). And, trivially, at time k, our portfolio will have the value E_k such that, at maturity, it replicates the claim. For the case with interest rates, all we needed to do was to define the *discounted* stock process $Z_i = B_i^{-1} S_i$ and use this process Z to find the martingale measure $\mathbb{Q}$. And we would consider the discounted claim to define the process E via $E_i = \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X|\mathcal{F}_i)$. The value of the claim X at time-tick i was then given by

$$V_i = B_i E_i = B_i \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X|\mathcal{F}_i)\,. \tag{1}$$

All these ideas carry over to the continuous world. Again, let us first write down the continuous version of the martingale representation theorem:

Theorem. (Martingale representation theorem)
Given a $\mathbb{Q}$-martingale M whose volatility is always non-zero and any other $\mathbb{Q}$-martingale N, there exists a previsible process ϕ such that N can be written as

$$N_t = N_0 + \int_0^t \phi_s \, dM_s$$

Using this theorem, we can proceed in the continuous world exactly in the same way as in the discrete world: First, we find a measure $\mathbb{Q}$, such that the stock process S_t is a $\mathbb{Q}$-martingale (again, we have $r = 0$). Then, convert the claim X into a process via $E_t = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t)$. Now, apply the martingale representation theorem to construct a previsible process ϕ_t such that

$$dE_t = \phi_t \, dS_t, \qquad E_t = E_0 + \int_0^t \phi_s dS_s$$

Again, the hedging strategy will consist in holding ϕ_t units of the stock at time t and $\psi_t = E_t - \phi_t S_t$ units of the bond, such that the value of the portfolio at time t will be $V_t = \phi_t S_t + \psi_t$. What happens if we have an interest rate $r > 0$? Similar to the discrete world, we need to consider now the *discounted* stock process $Z_t = B_t^{-1} S_t$ and use this process to find the martingale measure $\mathbb{Q}$. This is exactly what was done as an example at the end of the last section for the Black-Scholes model. The value of the claim X at time t is then - compare to (1)

$$V_t = B_t E_t = B_t \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t) \,. \tag{2}$$

For the standard bond process $B_t = B_0 \mathrm{e}^{rt}$ we find

$$V_t = B_0 \mathrm{e}^{rt} \mathbb{E}_{\mathbb{Q}}(B_0^{-1} \mathrm{e}^{-rT} X | \mathcal{F}_t) = \mathrm{e}^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_t) \tag{3}$$

The Black-Scholes Model

We already noted in the section introducing the change of measure as an example how to derive the Black-Scholes formula using the Cameron-Martin-Girsanov theorem.

$$\begin{aligned} V &= \mathrm{e}^{-rT} \mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{Q}} \left(\left(S_0 \, \mathrm{e}^{\sigma \tilde{W}_t - \sigma^2 t/2} - K \mathrm{e}^{-rT} \right)^+ \right) \\ &= S_0 \Phi \left(\frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}} \right) \\ & \quad - K \, \mathrm{e}^{-rT} \, \Phi \left(\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} \right) \end{aligned}$$

Clearly, (if the claim X, as always assumed only depends on the final value of the stock S_T), for a time $0 < t < T$, we can write the value $V_t = V(S_t, t)$ for $s = S_t$ as

$$V(s,t) = \mathrm{e}^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}(X | S_t = s) \tag{4}$$

and, concerning the trading strategy, the number of shares that one needs to hold to hedge the claim at time t is given by $\phi_t = \partial V / \partial s$ which is the continuous version of $\phi = (f_u - f_d)/(s_u - s_d)$ valid in the discrete world. With a little bit of algebra, one can prove that

$$\phi_t = \frac{\partial V}{\partial s}(S_t, T - t) = \Phi \left(\frac{\ln(S_0/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{(T - t)}} \right) \tag{5}$$

A partial differential equation for the option price: A second proof for the formula $\phi_t = \partial V/\partial s$ comes from the fact that the hedging strategy is self-financing, meaning that we have

$$dV_t = \phi_t dS_t + \psi_t dB_t \tag{6}$$

Under the martingale measure, the SDE for the stock process yields

$$dS_t = \sigma S_t d\tilde{W}_t + rS_t dt \tag{7}$$

and, for the bond process, we have as always $dB_t = rB_t dt$. Together, this yields the following representation of dV_t:

$$dV_t = \sigma S_t \phi_t d\tilde{W}_t + (rS_t\phi_t + r\psi_t B_t)\, dt \tag{8}$$

Now we can derive a second representation of dV_t using Ito's lemma:

$$dV_t = dV(S_t,t) = \left(\sigma S_t \frac{\partial V}{\partial s}\right) d\tilde{W}_t + \left(rS_t\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial s^2} + \frac{\partial V}{\partial t}\right) dt \tag{9}$$

Comparing the two expressions, we find

$$\phi_t = \frac{\partial V}{\partial s}, \qquad rs\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + \frac{\partial V}{\partial t} = rV \tag{10}$$

The latter equation is a *partial differential equation* that can be solved (with the appropriate final condition) in order to obtain the option price.
Proof of the hedging formula: We will now prove the hedging formula given by (5). This is actually not more than an exercise in taking derivatives correctly, but it is worthwhile as the result is important. First, we show that Show that $s\Phi'(d_1) = k\mathrm{e}^{-r(T-t)}\Phi'(d_2)$ when d_1 and d_2 are defined as

$$\begin{aligned} d_1 &= \frac{\log(s/k) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log(s/k) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \end{aligned}$$

This part is pure algebra: Set $T-t = \tau$, and we need to show that $s\Phi'(d_1) = k\mathrm{e}^{-r\tau}\Phi'(d_2)$, so we use the Gaussian density:

$$s\exp\left(-\frac{1}{2}\left(\frac{\ln(s/k) + (r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)^2\right) = k\mathrm{e}^{-r\tau}\exp\left(-\frac{1}{2}\left(\frac{\ln(s/k) + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)^2\right).$$

Using now $\exp(-\ln(s/k)) = k/s$, we see that this statement is equivalent to

$$-\frac{1}{2\sigma^2\tau}\left(\ln(s/k) + (r+\sigma^2/2)\tau\right)^2 = -\ln(s/k) - r\tau - \frac{1}{2\sigma^2\tau}\left(\ln(s/k) + (r-\sigma^2/2)\tau\right)^2 ,$$

an expression that simplifies immediately when resolving the squares:

$$\begin{aligned}
&-\frac{1}{\sigma^2\tau}\ln(s/k)(r+\sigma^2/2)\tau - \frac{1}{2\sigma^2\tau}(r+\sigma^2/2)^2\tau^2 \\
= &-\ln(s/k) - r\tau - \frac{1}{\sigma^2\tau}\ln(s/k)(r-\sigma^2/2)\tau - \frac{1}{2\sigma^2\tau}(r-\sigma^2/2)^2\tau^2
\end{aligned}$$

and, looking at this expression twice, you'll see that this is correct! Here is why this is useful: Consider the value of a European option in the Black-Scholes model at time t and set $\tau = T - t$. Then we know that this value is given by the Black-Scholes formula in the form

$$V(s,\tau) = s\Phi(d_1(\tau)) - k\mathrm{e}^{-r\tau}\Phi(d_2(\tau))$$

and the hedging is given by

$$\frac{\partial V}{\partial s} = \Phi(d_1(\tau)) + s \cdot \frac{1}{s\sigma\sqrt{\tau}}\Phi'(d_1(\tau)) - \frac{k\mathrm{e}^{-r\tau}}{s\sigma\sqrt{\tau}}\Phi'(d_2(\tau)) = \Phi(d_1(\tau))\,.$$

One more example: We conclude this section with another example of pricing a different option using the Black-Scholes model: Consider a stock $S_t = S_0\mathrm{e}^{\sigma W_t}$ and assume $\sigma = 0.2$, $S_0 = \$10$ and that interest rates are zero. What is the value of a bet that pays \$20 if the stock is worth less than \$8 after two years? The starting point is to construct the risk-neutral measure $\mathbb{Q}$ such that S_t becomes a martingale. We know that the drift can be eliminated using the Girsanov theorem by constructing a measure $\mathbb{Q}$ such that $d\tilde{W}_t = dW_t + \gamma dt$ with $\gamma = \sigma/2$, where $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion. In terms of $\tilde{W}_t$ we have

$$S_t = S_0\,\mathrm{e}^{\sigma\tilde{W}_t - \sigma^2 t/2}\,.$$

To price the claim, we need to compute

$$V = \mathbb{E}_{\mathbb{Q}}(X) = 20 \cdot P_{\mathbb{Q}}(S_T < 8)$$

Here, we denote by $P_{\mathbb{Q}}(S_T < 8)$ the *risk-neutral* probability that the stock price S_T takes a value of less than \$8. We can express this probability as

$$P_{\mathbb{Q}}(S_T < 8) = \int_{-\infty}^{z_c} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}\,dz$$

where the upper limit z_c is given by

$$10 \cdot \mathrm{e}^{\sigma\sqrt{T}z_c - \sigma^2 T/2} = 8, \qquad z_c = \frac{\ln(8/10) + \sigma^2 T/2}{\sigma\sqrt{T}}\,.$$

Hence the price of the derivative results in

$$V = 20 \cdot \Phi\left(\frac{\ln(8/10) + \sigma^2 T/2}{\sigma\sqrt{T}}\right) \approx 20 \cdot \Phi(-0.64751) \approx 5.173$$