Change of Measure (Cameron-Martin-Girsanov Theorem)

Radon-Nikodym derivative: Taking again our intuition from the discrete world, we know that, in the context of option pricing, we need to price the claim using the risk-neutral measure. If there are no interest rates, this measure $\mathbb{Q}$ is constructed through the requirement that the stock process S_t needs to be a $\mathbb{Q}$-martingale. This will also work in the continuous world, however, we will need a continuous model of the stock process in the first place. For the Black-Scholes model, we start from

$$S_t = S_0 \mathrm{e}^{\sigma W_t + \mu t}$$

with a $\mathbb{P}$-Brownian motion W_t. In order to define a measure $\mathbb{Q}$, such that S_t becomes a $\mathbb{Q}$-martingale, we need to know how a Brownian motion changes when the measure changes. Or, in other words, we would like to express the change of measure in terms of Brownian motions.

To prepare the change of measure in the continuous world, we go back (only for a moment) to the discrete world. Consider the same tree with two different measures $\mathbb{P}$ and $\mathbb{Q}$.

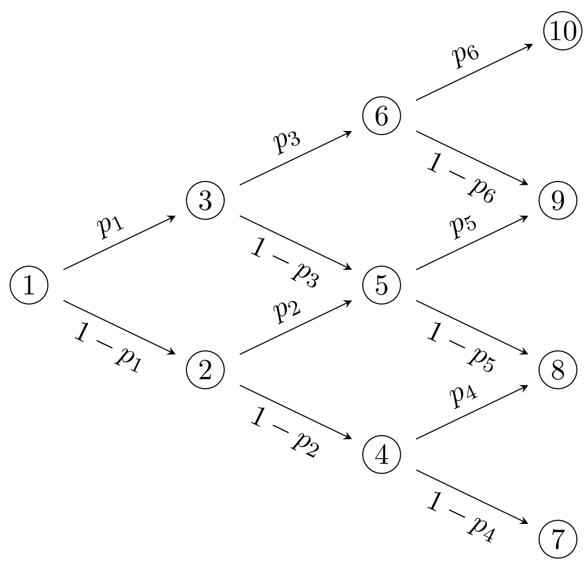

Figure 1: Binomial tree with measure $\mathbb{P}$

In the following, we basically only need the rule that we have to multiply probabilities on branches in order to compute the probability to get to a

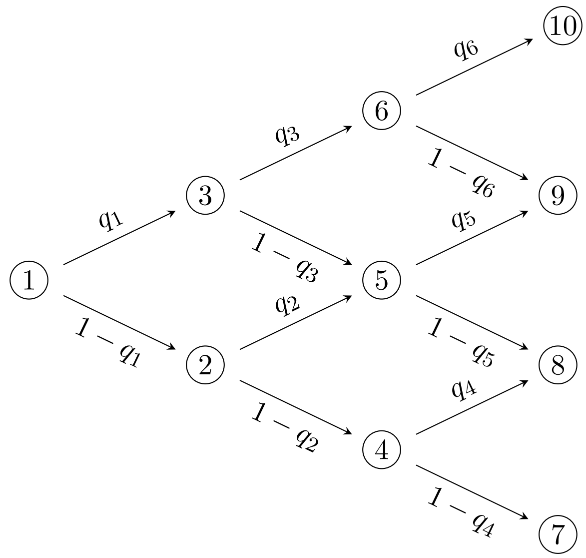

Figure 2: Binomial tree with measure $\mathbb{Q}$

certain node. For instance, using the measure $\mathbb{P}$, the probability π_{10} to get to the node numbered node 10 is

$$\pi_{10} = p_1 \cdot p_3 \cdot p_6$$

Under the measure $\mathbb{Q}$, the probability to reach node 9 is in general different. Let's denote it by $\tilde{\pi}_{10}$. Obviously, we have

$$\tilde{\pi}_{10} = q_1 \cdot q_3 \cdot q_6$$

In this way, we can express probabilities for all nodes. For node 9, we find

$$\begin{aligned} \pi_9 &= p_1 \cdot p_3 \cdot (1 - p_6) + p_1 \cdot (1 - p_3) \cdot p_5 + (1 - p_1) \cdot p_2 \cdot p_5, \\ \tilde{\pi}_9 &= q_1 \cdot q_3 \cdot (1 - q_6) + q_1 \cdot (1 - q_3) \cdot q_5 + (1 - q_1) \cdot q_2 \cdot q_5 \,. \end{aligned}$$

After these preparations, we are ready to think about the question under which circumstances it is possible to express the measure $\mathbb{Q}$ in terms of the measure $\mathbb{P}$ and vice versa. The (trivial) answer is to write simply for the probability to reach node 9

$$\tilde{\pi}_9 = \frac{\tilde{\pi}_9}{\pi_9} \cdot \pi_9,$$

in order to go from measure $\mathbb{P}$ to the measure $\mathbb{Q}$ and, on the other hand,

$$\pi_9 = \frac{\pi_9}{\tilde{\pi}_9} \cdot \tilde{\pi}_9 \,.$$

in order to go from measure $\mathbb{Q}$ to the measure $\mathbb{P}$. First, we notice that the likelihood ratio corresponds to a stochastic process. This process is called *Radon-Nikodym derivative* and denoted by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ or $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}$. Clearly, for the Radon-Nikodym derivative to be well-defined, we need to assume that nodes of the tree that are accessible under the measure $\mathbb{Q}$ are also accessible under the measure $\mathbb{P}$. In other words: we need to avoid dividing by zero when forming the likelihood ratios. The formal definition is given by the *equivalence* of the two measures: The two measures are equivalent if for each set A the statement $\mathbb{Q}(A) > 0$ is equivalent to $\mathbb{P}(A) > 0$. For the binomial tree, this reduces to the statement that, nodes that are accessible under one measure are also accessible under the other measure. By 'accessible' we mean simply that there is a non-zero probability to reach a node.

For equivalent measures, we can easily express the expectation value with respect to one measure through the expectation value taken with respect to the other measure. Consider a discrete random variable X, we find

$$\mathbb{E}_{\mathbb{P}}(X) = \sum_i x_i \pi_i = \sum_i x_i \frac{\pi_i}{\tilde{\pi}_i} \tilde{\pi}_i = \mathbb{E}_{\mathbb{Q}} \left(X \frac{d\mathbb{P}}{d\mathbb{Q}} \right) \tag{1}$$

For a normally distributed random variable, we can use this formula to characterize the change of measure. This is due to the following basic fact:

Theorem. (Characterization of Gaussian variables)
The following two statements are equivalent:

1. *A random variable X has a normal distribution $N(\mu, \sigma^2)$ under a measure $\mathbb{P}$*

2. *For all real θ, we have*

$$\mathbb{E}_{\mathbb{P}} \left(\mathrm{e}^{\theta X} \right) = \mathrm{e}^{\theta\mu + \theta^2\sigma^2/2}$$

Let's look at a simple, but very instructive example: In order to define a suitable Radon-Nikodym derivative, a simple choice is to have $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} > 0$. Let W_t be a $\mathbb{P}$-Brownian motion (hence normally distributed with zero mean and a variance t), a possible choice would be

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathrm{e}^{-\gamma W_t - \gamma^2 t/2} > 0 \,.$$

Can we figure out what happens to W_t under the measure $\mathbb{Q}$? If we are lucky, W has still a normal distribution under the new measure (but maybe with a different mean and/or variance). In this case, we should be able to use the above theorem. Let's try to compute

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(\mathrm{e}^{\theta W_t}\right) &= \mathbb{E}_{\mathbb{P}}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\mathrm{e}^{\theta W_t}\right) = \mathbb{E}_{\mathbb{P}}\left(\mathrm{e}^{-\gamma W_t-\gamma^2 t/2}\mathrm{e}^{\theta W_t}\right) \\
&= \mathrm{e}^{-\gamma^2 t/2}\mathbb{E}_{\mathbb{P}}\left(\mathrm{e}^{(\theta-\gamma)W_t}\right) = \mathrm{e}^{-\gamma^2 t/2}\mathrm{e}^{(\theta-\gamma)^2 t/2} \\
&= \mathrm{e}^{-\gamma\theta t+\theta^2 t/2}
\end{aligned}$$

Applying the above theorem again, we see that W has, under the measure $\mathbb{Q}$, again a normal distribution with mean $-\gamma t$ and variance t. In summary, we have shown that, for a $\mathbb{P}$ Brownian motion W_t, we can construct an equivalent measure $\mathbb{Q}$ such that, under the new measure $\mathbb{Q}$, the Brownian motion W_t has a mean $-\gamma t$ and a drift t. In particular $\tilde{W}_t = W_t + \gamma t$ is actually a $\mathbb{Q}$-Brownian motion, since the term γt is used to compensate for the negative drift. This is a special case of the following theorem:

Theorem. (Cameron-Martin-Girsanov theorem)
For a $\mathbb{P}$-Brownian motion W_t and a previsible process γ_t, satisfying the condition

$$\mathbb{E}_{\mathbb{P}}\left(\exp\left(\frac{1}{2}\int_0^T \gamma_t^2\,dt\right)\right) < \infty$$

there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that

$$\tilde{W}_t = W_t + \int_0^t \gamma_s\,ds$$

is a $\mathbb{Q}$-Brownian motion. The measures are related by the Radon-Nikodym derivative given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^t \gamma_s\,dW_s - \frac{1}{2}\int_0^t \gamma_s^2\,ds\right).$$

In the context of derivative pricing, we can use the Cameron-Martin-Girsanov theorem in order to construct a martingale measure. Consider for example a stochastic process $X_t = \mu t + \sigma W_t$ with a $\mathbb{P}$-Brownian motion W_t. Now, we can write

$$X_t = \sigma\left(\frac{\mu}{\sigma}t + W_t\right) = \sigma(\gamma t + W_t) = \sigma\tilde{W}_t$$

since the Cameron-Martin-Girsanov theorem guarantees the existence of an equivalent measure $\mathbb{Q}$ and the corresponding $\mathbb{Q}$-Brownian motion $\tilde{W}_t$. Clearly, X_t is not a $\mathbb{P}$-martingale, but it is obviously a $\mathbb{Q}$-martingale.

Derivation of the Black-Scholes formula using the Girsanov theorem: We can now derive again the Black-Scholes formula using the Cameron-Martin-Girsanov theorem. The main step consists in considering the Black-Scholes model with a stock and bond process given by

$$S_t = S_0\,\mathrm{e}^{\mu t + \sigma W_t}, \qquad B_t = B_0 \mathrm{e}^{rt}$$

and forming the discounted stock process $Z_t = B_t^{-1} S_t = S_0\,\mathrm{e}^{(\mu - r)t + \sigma W_t}$. For a European claim X with maturity T, the initial price is given by $V = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}(X)$, and Z_t is a $\mathbb{Q}$-martingale. Clearly, we want to use the Cameron-Martin-Girsanov theorem to construct $\mathbb{Q}$. Therefore, we use first Ito's lemma to find dZ_t:

$$dZ_t = Z_t\left((\mu - r + \sigma^2/2)\,dt + \sigma dW_t\right) = \sigma Z_t\left(\gamma\,dt + dW_t\right)\,, \tag{2}$$

where we set $\gamma = \mu - r + \sigma^2/2$. Applying now the Cameron-Martin-Girsanov theorem, we can construct the measure $\mathbb{Q}$ such that Z_t becomes a $\mathbb{Q}$-martingale and $\tilde{W}_t = \gamma t + W_t$ is a $\mathbb{Q}$-Brownian motion. Clearly, we have then $W_t = \tilde{W}_t - \gamma t$ and we can use this in order to express the stock process S_t in terms of $\tilde{W}_t$:

$$S_t = S_0\,\mathrm{e}^{\mu t + \sigma W_t} = S_0\,\mathrm{e}^{\mu t + \sigma \tilde{W}_t - (\mu - r)t - \sigma^2 t/2} = S_0\,\mathrm{e}^{\sigma \tilde{W}_t + (r - \sigma^2/2)t} \tag{3}$$

For a European call, we have $X = (S_T - K)^+$, and thus the price of such a call is computed as

$$V = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{Q}}\left(\left(S_0\,\mathrm{e}^{\sigma\tilde{W}_t - \sigma^2 t/2} - K\mathrm{e}^{-rT}\right)^+\right) \tag{4}$$

which is exactly the Black-Scholes formula.