Random Walks and Brownian Motion

Simple random walk: In the following, we develop a more systematic approach to formulate a continuum limit for stochastic processes. For this purpose, imagine that we devide the interval $[0, 1]$ into n steps and define a *random walk* W_n at time steps $t_0 = 0$, $t_1 = \delta t$, $t_2 = 2\delta t$, ... with $\delta t = 1/n$ as the process that

- starts at zero, hence $W_n(0) = 0$,
- can go up or down by $1/\sqrt{n}$ with a probability of $1/2$ at each step

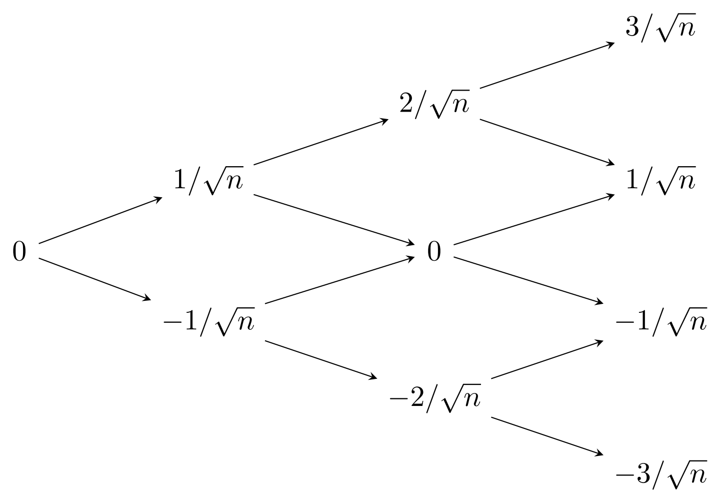

Figure 1: Simple random walk

The scaling of the 'up-' and 'down-' jumps of $1/\sqrt{n}$ is essential for the convergence of the process as we will see by the following argument: The random walk W_n can be written as a sum of independent random variables X_j, where each X_j takes the value -1 or 1 with probability 1/2, hence

$$W_n(k\delta t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} X_j \tag{1}$$

Clearly, the mean of W_n is zero. The variance at $k = n$ is computed as the sum of the variances of the X_j as all X_j are independent. Therefore, we find

$$\mathrm{Var}(W_n(1)) = \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^2 \mathrm{Var}(X_j) = n \cdot 1/n = 1 \tag{2}$$

Now we see that, due to the factor $1/\sqrt{n}$, the variance of $W_n(1) = 1$ for all n, hence we can expect convergence of $W_n \to Z$ with $Z \sim N(0,1)$ by applying the central limit theorem. Often, we also write

$$W_n((k+1)\delta t) = W_n(k\delta t) + \Delta W, \qquad \Delta W = \pm \frac{1}{\sqrt{n}} = \pm\sqrt{\delta t} \tag{3}$$

where ΔW is called the *Brownian increment* which can take the values $\sqrt{\delta t}$ and $-\sqrt{\delta t}$ with probability 1/2.

Brownian motion: We can now take the continuum limit to see that the random walk W_n converges to a continuous stochastic process W called *Brownian motion* (with respect to the measure $\mathbb{P}$. From the properties of W_n we can see that

- $W_0 = 0$
- $W_t \sim N(0,t)$
- $W_t - W_s \sim N(0, t-s)$

Moreover, due to the independence of the increments X_j, we know that all Brownian increments are independent, in particular $W_t - W_s$ is independent of the history up to the time s.

Brownian Motion as a Martingale

In continuous time, we define a martingale in analogy to the discrete definition: A stochastic process M_t is a martingale with respect to a measure $\mathbb{P}$ (or short $\mathbb{P}$-martingale) if for all $t > s$

$$\mathbb{E}_{\mathbb{P}}(M_t|\mathcal{F}_s) = M_s \tag{4}$$

It is easy to see that a $\mathbb{P}$-Brownian motion is a $\mathbb{P}$-martingale. Intuitively, this is clear as the random walk W_n goes up and down with the same probability. Using the above properties of Brownian motion, we can easily carry out a formal proof (assuming $t > s$):

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(W_t|\mathcal{F}_s) &= \mathbb{E}_{\mathbb{P}}(W_t - W_s + W_s|\mathcal{F}_s) \\
&= \mathbb{E}_{\mathbb{P}}(W_t - W_s|\mathcal{F}_s) + \mathbb{E}_{\mathbb{P}}(W_s|\mathcal{F}_s) \\
&= 0 + W_s = W_s
\end{aligned}$$

In the last step, we used the (trivial) fact that $\mathbb{E}_{\mathbb{P}}(W_s|\mathcal{F}_s) = W_s$ and that $\mathbb{E}_{\mathbb{P}}(W_t - W_s|\mathcal{F}_s) = 0$ since we know that $W_t - W_s \sim N(0, t-s)$. We can also use the properties of Brownian motion in order to carry out more complicated calculations. Consider for example the stochastic process

$$Y_t = W_t^2 - t$$

and we can show that Y_t is also a $\mathbb{P}$-martingale in the following way: First we show that

$$\mathbb{E}_{\mathbb{P}}(W_t^2 - W_s^2 | \mathcal{F}_s) = t - s \tag{5}$$

Proof: By direct computation we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(W_t^2 - W_s^2 | \mathcal{F}_s) &= \mathbb{E}_{\mathbb{P}}((W_t - W_s)^2 + 2W_s(W_t - W_s) | \mathcal{F}_s) \\
&= \mathbb{E}_{\mathbb{P}}((W_t - W_s)^2 | \mathcal{F}_s) + 2W_s \mathbb{E}_{\mathbb{P}}((W_t - W_s) | \mathcal{F}_s) \\
&= t - s\,.
\end{aligned}$$

Again, the last line follows directly from the fact that $W_t - W_s \sim N(0, t-s)$. With the above relation (**??**) at hand, we find for the process Y_t:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(W_t^2 - t | \mathcal{F}_s) &= \mathbb{E}_{\mathbb{P}}(W_t^2 - W_s^2 + W_s^2 - t | \mathcal{F}_s) \\
&= t - s + W_s^2 - t = W_s^2 - s
\end{aligned}$$

which shows that $Y_t = W_t^2 - t$ is indeed a martingale. This calculation, however, shows as well that it will be useful to find more sophisticated tests to see whether a process in continuous time is a martingale. We will see that the so-called Ito's Lemma (see below) offers for many processes a quick way to test whether they satisfy the martingale property.

Stochastic Differential Equations

Consider the process defined by $X_t = \sigma_t W_t$. In a discrete approximation, we can compute the next value $X_{t+\Delta t}$ by drawing a random number r which is 1 or -1 with probability 1/2 and setting

$$X_{t+\Delta t} = X_t + \sigma_t \, r \sqrt{\Delta t}\,.$$

A different way of writing this relationship is to consider the differential by writing

$$\Delta X_t = X_{t+\Delta t} - X_t = \sigma_t \, r \sqrt{\Delta t} = \sigma_t \, \Delta W$$

with the Brownian increment ΔW. The continuous version of the above equation is simply

$$dX_t = \sigma_t dW_t \tag{6}$$

and, in this form, the stochastic equation resembles a differential equation. *A (short) review of differential equations:* Let's, for a moment, put stochasticity aside and consider a basic, deterministic differential equation of the form

$$\frac{dB_t}{dt} = \mu_t$$

Writing this in a discretized version, we can write

$$\Delta B_t = B_{t+\Delta t} - B_t = \mu_t \, \Delta t$$

The continuous version of the above equation is simply

$$dB_t = \mu_t dt$$

which is very similar to (**??**). However, the main difference is that Δt is deterministic, whereas ΔW is stochastic. Moreover, the size of ΔW is much larger that Δt as $|\Delta W| = \sqrt{\Delta t}$.

General form of a stochastic differential equation: It is convenient to combine stochastic and deterministic contributions to the change of the stochastic process X_t in differential form by writing

$$dX_t = \mu_t dt + \sigma_t dW_t \tag{7}$$

Note that, as for ordinary differential equations the function μ_t and σ_t can depend on X_t and on t, hence

$$\mu_t = \mu(X_t, t), \qquad \sigma_t = \sigma(X_t, t)\,.$$

As for deterministic differential equations, there are no general techniques to solve any given stochastic differential equation. Many simple problems, however, can be solved by applying Ito's Lemma which we will discuss in the next section.

Ito's Lemma

Taylor expansion of the differential: Consider a stochastic differential equation in its general form

$$dX_t = \mu_t dt + \sigma_t dW_t$$

and consider a transformation of the form

$$Y_t = f(X_t)$$

with a smooth function f. We can calculate the differential equation for Y_t by considering the following Taylor expansion:

$$
\begin{aligned}
\Delta Y &= f(X_t + \Delta X) - f(X_t) = f'(X_t)\Delta X + \frac{1}{2} f''(X_t)(\Delta X)^2 + ... \\
&= f'(X_t)(\mu_t \Delta t + \sigma_t \Delta W) + \frac{1}{2} f''(X_t)(\mu_t \Delta t + \sigma_t \Delta W)^2 + ... \\
&\approx f'(X_t)(\mu_t \Delta t + \sigma \Delta W) + \frac{1}{2} f''(X_t)\, \sigma_t^2 (\Delta t)
\end{aligned}
$$

The last approximation follows as $(\Delta W)^2 = \Delta t$ and from the fact that we consider terms up to the order Δt. Remember that $|\Delta W| = \sqrt{\Delta t}$. Terms like $(\Delta W)\Delta t$ are of the order $(\Delta t)^{3/2}$ and can be neglected. Therefore, we have the following

Lemma. (Ito's Lemma) *Assume that a stochastic process X_t is the solution of the stochastic differential equation*

$$
dX_t = \mu_t dt + \sigma_t dW_t \,.
$$

Then, for a transformed process $Y_t = f(X_t)$ with a deterministic, twice continuously differentiable function f, the process Y_t satisfies the stochastic differential equation given by

$$
\begin{aligned}
dY_t &= f'(X_t)\left(\mu_t dt + \sigma_t dW_t\right) + \frac{1}{2}\sigma_t^2 f''(X_t) dt \\
&= f'(X_t)\sigma_t dW_t + \left(f'(X_t)\mu_t + \frac{1}{2}\sigma_t^2 f''(X_t) \right) dt \,.
\end{aligned}
$$

Examples of the application of Ito's lemma: We can consider Ito's lemma as a generalization of the deterministic chain rule. In the following we discuss several examples and applications of Ito's lemma.

- Consider the special case, where $X_t = W_t$ and $Y_t = f(W_t)$. Then we have $\mu_t = 0$ and $\sigma_t = 1$ and we see that

$$
dY_t = f'(W_t) dW_t + \frac{1}{2} f''(W_t) dt \,.
$$

- Applying the above formula, we find directly relationships like

$$
\begin{aligned}
d(W_t^2) &= 2W_t dW_t + dt \\
d(W_t^2 - t) &= 2W_t dW_t \\
d(W_t^3) &= 3W_t^2 dW_t + 3W_t dt \\
d(W_t^6) &= 6W_t dW_t + 15W_t^4 dt
\end{aligned}
$$

In particular we see that the solution of the stochastic differential equation $dY_t = 2Y_t dY_t$ (together with the initial condition $Y_0 = 0$) is given by $Y_t = W_t^2 - t$.

- An important example is the application to $f(x) = \mathrm{e}^x$. Then, clearly, $f(x) = f'(x) = f''(x)$. Let's assume constant volatility $\sigma_t = \sigma$ and constant drift $\mu_t = \mu$ and define

  $$Y_t = f(X_t) = \mathrm{e}^{X_t} = \mathrm{e}^{\sigma W_t + \mu t}\,.$$

  This process is called *exponential Brownian motion* and is of relevance in the context of the Black-Scholes model. Applying Ito's lemma, we find that

  $$\begin{aligned} dY_t &= f'(X_t)dX_t + \frac{1}{2}\sigma^2 f''(X_t)dt \\ &= Y_t\left(\sigma dW_t + \left(\mu + \frac{1}{2}\sigma^2\right)dt\right)\,. \end{aligned}$$

  In particular we see that for $\mu = -\sigma/2$, hence for

  $$Y_t = \mathrm{e}^{\sigma W_t - \sigma^2 t/2}$$

  we have

  $$dY_t = \sigma Y_t dW_t$$

- For two processes X_t and Y_t satisfying the stochastic differential equations

  $$\begin{aligned} dX_t &= \mu_t dt + \sigma_t dW_t \\ dY_t &= \nu_t dt + \rho_t dW_t \end{aligned}$$

  we have for the product $Z_t = X_t Y_t$

  $$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \sigma_t \rho_t dt\,.$$

  This generalizes the product rule to stochastic processes. The proof is left as an exercise. Hint: use a two-dimensional Taylor expansion of the function $f(x,y) = xy$.

Using Ito's lemma to identify martingales: Aside from solving stochastic differential equations, we can also use Ito's lemma to identify martingales.

To see why, let's prove first the following (trivial) statement: Consider a stochastic process X_t given by

$$X_t = \sigma W_t + \mu t$$

with constant volatility σ and constant drift μ. Then X_t is a martingale if, and only if, the drift vanishes or $\mu = 0$. The proof is basically a repetition of the proof that Brownian motion is a martingale: Assume X_t is a martingale, then we have (for any $t > s$)

$$\mathbb{E}_{\mathbb{P}}(X_t|\mathcal{F}_s) = X_s \, .$$

As $X_s = \sigma W_s + \mu s$, and using the fact that W_t is a $\mathbb{P}$-martingale, we see that

$$\mathbb{E}_{\mathbb{P}}(X_t|\mathcal{F}_s) = \sigma \mathbb{E}_{\mathbb{P}}(W_t|\mathcal{F}_s) + \mu t = \sigma W_s + \mu t = X_s = \sigma W_s + \mu s$$

or $\mu t = \mu s$, hence $\mu = 0$. The other direction is trivial. The important interpretation of this statement is that, for *arithmetic Brownian motion* $\sigma W_t + \mu t$, we need the drift to vanish in order for the process to be a martingale. This statement is generalized by the following theorem:

Theorem. (Characterization of martingales) *Assume that a stochastic process X_t is the solution of the stochastic differential equation*

$$dX_t = \mu_t dt + \sigma_t dW_t \, .$$

and that the (technical) condition $\mathbb{E}((\int_0^T \sigma_s^2 ds)^{1/2}) < \infty$ is satisfied. Then X_t is a martingale if, and only if, X_t is driftless (hence $\mu_t = 0$).

We will not give a formal proof of this theorem, but taking our intuition from the discrete world, we know that a process on a finite tree is a martingale if the process is a martingale on each branch. Therefore, for a time-tick Δt, we can consider

$$X_{t+\Delta t} = X_t + \Delta X = \begin{cases} \mu_t \Delta t + \sigma_t \sqrt{\Delta t} \\ \mu_t \Delta t - \sigma_t \sqrt{\Delta t} \end{cases}$$

where the probability for an 'up'-jump and an 'down'-jump are 1/2. Therefore, we have

$$\mathbb{E}_{\mathbb{P}}(X_{t+\Delta t}|\mathcal{F}_t) = X_t + \mu_t \Delta t = X_t$$

if X_t is a $\mathbb{P}$-martingale. Hence $\mu_t = 0$ for that time-tick. This local feature of one 'branch' translates to the entire tree, hence $\mu_t = 0$ everywhere.

Going back to one of the examples for Ito's lemma discussed earlier, we can now see immediately that the stochastic process $Y_t = W_t^2 - t$ is a martingale as there is no drift present in $dY_t = 2W_t dW_t$. On the other hand, $Y_t = W_t^2$ is not a martingale since the stochastic differential equation is $dY_t = 2W_t dW_t + dt$ with the drift term dt present. We actually already obtained this result above directly from the properties of Brownian motion. The calculation, however, was more tedious than a simple and direct application of the above theorem in conjunction with Ito's lemma.

In the context of stock models, referring to the exponential Brownian motion we see that

$$Y_t = Y_0 \mathrm{e}^{\sigma W_t - \sigma^2 t/2}$$

is a martingale as there is no drift term in the corresponding stochastic differential equation

$$dY_t = \sigma Y_t dW_t \, .$$

This already resembles the representation of the stock process in the martingale measure that we obtained earlier for the Black-Scholes model. The precise meaning will become clear in the next section when we discuss the change of measure and the Girsanov theorem.